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THE OVERALL RESPONSE OF COMPOSITE MATERIALS UNDERGOING LARGE DEFORMATIONS

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Abstract

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Research goals

The central goal of this project is to estimate the *effective* constitutive properties of *nonlinear* composite materials undergoing *large* deformations. Two types of large deformations are of particular interest: large *elastic* deformations, corresponding to materials such as polymeric composites, rubber foams and solid rocket fuel composites; and large *viscous* deformations, corresponding to the high-temperature creeping, or to the dynamic plastic deformation of metals. The main current goal—and the goal for the next funding period—will be to estimate the effective behavior of nonlinear *anisotropic* composites, such as fiber-reinforced/metal-matrix composites. Preliminary results (mostly for laminated materials) indicate that there may be significant coupling between different loading modes.

BACKGROUND

There is a fairly recent, but growing literature in the field of composite materials that is concerned with the prediction of the effective behavior of nonlinear composites. To date most of the work on the subject has made use of *ad hoc* models for certain classes of composites, as well as of heavy numerical computations for composites with periodic microstructures. For example, Duva and Hutchinson [7] determined *dilute* estimates for porous creeping composites by making use of the results of Budiansky, Hutchinson and Slutsky [3] for a single void in an infinite matrix of power-law viscous material. Duva [6] proposed a *differential self-consistent* estimate to model the reinforcing effect of rigid particles in a power-law creeping matrix. More recently, Bao, Hutchinson and McMeeking [2] proposed *generalized self-consistent* estimates for the yield stress of two-phase, rigid/perfectly plastic composites. On the other hand, Christman, Needleman and Suresh [4] and Bao, Hutchinson and McMeeking [1] carried out computations on the unit cell of a *periodic* microstructure to model the effect of particle size and shape on the effective behavior of rigidly reinforced power-law, metal-matrix composites.

As far as rigorous methods for estimating the effective behavior of composites with *random* microstructures (real composite materials are hardly periodic), the only method available until recently was the Talbot and Willis [15] extension of the Hashin-Shtrikman variational principles to composites with nonlinear behavior. Ponte Castañeda and Willis [14] applied this procedure to nonlinearly viscous materials, and gave the first rigorous bounds and self-consistent estimates for the effective properties of composites in power-law creep. Later, Ponte Castañeda [8] developed corresponding bounds in the context of finite elasticity. Very recently, under the sponsorship of AFOSR, Ponte Castañeda [9] developed a new general method for estimating and bounding the effective behavior of nonlinear composites. The method expresses the effective behavior of nonlinear composites in trms of an optimization problem involving the effective behavior of linear composites materials with similar microstructures. This is very useful because of the large body of results in the form of bounds and estimates of different types that are currently available for lineart composites. This method and some of its applications to specific types of composites are discussed in the following section.

ACCOMPLISHMENTS

Introduction

In this section, we give a brief account of the theoretical foundations of the new variational method developed to predict the effective behavior of nonlinear composites. Explicit results for composites with overall isotropy are given in references [9, 10], and corresponding results for fiber-reinforced and laminated microstructures in [11] and [5, 12], respectively. In particular, we note that our results for simple two-phase, power-law creep composites compare very favorably with the results of other authors obtained by means of sophisticated large-scale computations for periodic composites. Our results also have the advantage that they can often be given simple analytical forms. For example, we have been able to obtain an improved characterization of porous isotropic and transversely isotropic materials; in fact, because of the bounding properties of our results, we are able to show that the commonly used Gurson model is not appropriate outside the range of dilute porosity and high-triaxiality for which it was originally developed [13]. Similarly, we have been able to obtain yield surfaces for isotropic and transversely isotropic composites with rigid/perfectly plastic phases. For the benefit of the reader, some of the above mentioned references [5, 10, 11, 12] are included in the Appendix.

The method

Effective properties

Consider a two-phase heterogeneous material occupying a region in space of unit volume Ω. We characterize the local stress potential $\phi(\boldsymbol{\sigma},\mathbf{x})$ of the material in terms of the homogeneous phase potentials $\phi^{(r)}(\boldsymbol{\sigma})$ via the relation

$$\phi(\boldsymbol{\sigma},\mathbf{x}) = \sum_{r=1}^{2} \chi^{(r)}(\mathbf{x})\,\phi^{(r)}(\boldsymbol{\sigma}), \tag{1}$$

where the $\chi^{(r)}$ ($r = 1, 2$) are the characteristic functions of the two phases. We will assume that the two phases are nonlinear, incompressible and isotropic, so that the potentials $\phi^{(r)}(\boldsymbol{\sigma})$ depend only on the effective stress $\sigma_e = \sqrt{\frac{3}{2}\mathbf{S}\cdot\mathbf{S}}$, where $\mathbf{S}$ is the deviator of $\boldsymbol{\sigma}$. Thus, we write

$$\phi^{(r)}(\boldsymbol{\sigma}) = f^{(r)}(\sigma_e), \tag{2}$$

where the scalar-valued functions $f^{(r)}$ are assumed to satisfy some rather weak hypothesis. For example, $f^{(r)}$ may be chosen to have the power-law form

$$f(\sigma_e) = \frac{1}{n+1}\dot{\varepsilon}_o\sigma_o\left(\frac{\sigma_e}{\sigma_o}\right)^{n+1} \tag{3}$$

with n, σ_o and $\dot{\varepsilon}_o$ denoting material constants with different values in each phase.

The local constitutive relation for the creeping material is given by

$$\dot{\boldsymbol{\varepsilon}} = \frac{\partial\phi}{\partial\boldsymbol{\sigma}}(\boldsymbol{\sigma},\mathbf{x}), \tag{4}$$

where $\dot{\boldsymbol{\varepsilon}}$ is the rate-of-deformation (strain-rate) tensor.

Letting $\boldsymbol{\Sigma}$ and $\dot{\mathbf{E}}$ denote the average values of the stress and strain-rate fields, respectively, the effective, or overall, behavior of the heterogeneous creeping material is given by the relation

$$\mathbf{E} = \frac{\partial\Phi}{\partial\boldsymbol{\Sigma}}, \tag{5}$$

where the effective energy of the composite Φ follows from the principle of minimum complementary energy, which can be stated in the form

$$\Phi(\boldsymbol{\Sigma}) = \min_{\boldsymbol{\sigma}\in S(\boldsymbol{\Sigma})}\int_\Omega \phi(\boldsymbol{\sigma},\mathbf{x})\,dx, \tag{6}$$

where $S(\boldsymbol{\Sigma}) = \{\boldsymbol{\sigma}\,|\,\sigma_{ij,j} = 0 \text{ in } \Omega, \text{ and } \sigma_{ij}n_j = \Sigma_{ij}n_j \text{ on } \partial\Omega\}$ is the set of statically admissible stresses.

In this work, we are interested in predicting the effective behavior of two-phase heterogeneous materials, as described above. We further restrict our consideration to composites with phases in given volume fractions

$$c^{(r)} = \int_\Omega \chi^{(r)}(\mathbf{x})\,dx, \tag{7}$$

New variational principles

The new variational principles [9, 11] are based on a representation of the potential for the nonlinear isotropic material in terms of the potentials of a family of linear *comparison* materials. This representation is obtained with the help of the Legendre transformation. Thus, for a homogeneous nonlinear material with "superquadratic" growth in its potential ϕ, and certain additional convexity hypothesis, we have that

$$\phi(\boldsymbol{\sigma},\mathbf{x}) = \max_{\mu>0}\{\phi_o(\boldsymbol{\sigma},\mathbf{x}) - V(\mu,\mathbf{x})\}, \tag{8}$$

where

$$V(\mu,\mathbf{x}) = \max_{\boldsymbol{\sigma}}\{\phi_o(\boldsymbol{\sigma},\mathbf{x}) - \phi(\boldsymbol{\sigma},\mathbf{x})\} \tag{9}$$

and where ϕ_o is the potential of a linear comparison material with shear modulus μ. Note that in the above optimizations, the position vector $\mathbf{x}$ is fixed.

The new variational principle is obtained by averaging the relation (8) and the complementary energy principle (6) , and can be expressed in the form

$$\Phi(\boldsymbol{\Sigma}) = \max_{\mu(\mathbf{x})\geq 0}\left\{\Phi_o(\boldsymbol{\Sigma}) - \int_{\Omega} V(\mu,\mathbf{x})\,dx\right\}, \tag{10}$$

where

$$\Phi_o(\boldsymbol{\Sigma}) = \min_{\boldsymbol{\sigma}\in S(\boldsymbol{\Sigma})}\int_{\Omega}\phi_o(\boldsymbol{\sigma},\mathbf{x})\,dx \tag{11}$$

is the effective potential of a linear heterogeneous comparison material with local potential ϕ_o and arbitrarily variable shear moduli $\mu(\mathbf{x})$.

The variational principle described by (10) and (11) can be given the interpretation of first solving a linear problem for an arbitrarily heterogeneous *linear* material, and then optimizing with respect to the variations in moduli to account for the nonlinearity in the actual material. This suggests that if the fields happen to be constant over the nonlinear phases in the actual composite, then the variable moduli $\mu(\mathbf{x})$ can be replaced by constant moduli over each phase, *i.e.* by $\mu^{(1)}$ and $\mu^{(2)}$ in phase 1 and phase 2, respectively. More generally, however, we have the following lower bound for Φ, namely,

$$\Phi_{-}(\boldsymbol{\Sigma}) = \max_{\mu^{(1)},\mu^{(2)}\geq 0}\{\Phi_o(\boldsymbol{\Sigma}) - c^{(1)}V^{(1)}(\mu^{(1)}) - c^{(2)}V^{(2)}(\mu^{(2)})\}, \tag{12}$$

where $V^{(1)}$ and $V^{(2)}$ correspond to relation (9) evaluated for each of the homogeneous phase potentials $\phi^{(1)}$ and $\phi^{(2)}$, and Φ_o is the effective energy of an isotropic two-phase linear composite with phase moduli $\mu^{(1)}$ and $\mu^{(2)}$ in volume fractions $c^{(1)}$and $c^{(2)}$, respectively. This latter expression has the advantage over the previous more general relation that it involves only a finite-dimensional optimization problem, and can be further used in conjunction with bounds and estimates for the linear composite to induce corresponding results for the nonlinear composite. We illustrate these possibilities in the next section for a general two-phase incompressible composite.

Sample application to two-phase creeping systems

The estimate (12) of the previous section for the effective energy of the nonlinear incompressible composite can be used in conjunction with the well-known Hashin-Shtrikman bounds for the effective properties of the linear comparison composite to yield estimates for the effective potential of the nonlinear composite. It is shown in reference [11] that the result of this calculation takes on the simple form

$$\Phi_{HS}(\boldsymbol{\Sigma}) = \min_{\omega \geq 0} \left\{ c^{(1)} f^{(1)}(S^{(1)}) + c^{(2)} f^{(2)}(S^{(2)}) \right\}, \tag{13}$$

where $f^{(1)}$ and $f^{(2)}$ characterize the phase behaviors, and $S^{(1)}$ and $S^{(2)}$ are simple expressions of ω involving the phase volume fractions $c^{(1)}$ and $c^{(2)}$, and the overall effective stress applied to the composite Σ_e. For instance, if $\mu^{(1)} \geq \mu^{(2)}$ and the Hashin-Shtrikman lower bound is used for Φ_o, the corresponding expressions for $S^{(1)}$ and $S^{(2)}$ become

$$S^{(1)} = \sqrt{\left(1 - c^{(2)}\omega\right)^2 + \frac{2}{3} c^{(2)} \omega^2}\, \Sigma_e \quad \text{and} \quad S^{(2)} = \left(1 + c^{(1)}\omega\right)\Sigma_e \tag{14}$$

Depending on the relative strengths of the two nonlinear potentials $f^{(1)}$ and $f^{(2)}$, the result of putting (14) into (13) would either lead to a rigorous lower bound for the nonlinear potential Φ, or alternatively, to an upper "estimate" for Φ. Other examples are detailed in the Appendices.

RESEARCH PLANS FOR THE FUTURE

In the remaining nine months of support of this project, we plan to complete the development of effective constitutive relations for *anisotropic* nonlinear composites, including fiber-reinforced solids, for a fairly broad range of constitutive behaviors for the matrix and inclusion phases. We also plan to get started in the study of damage localizations in composites, which may arise because of the finite kinematics, or alternatively through thermomechanical, or other micro-mechanisms. These are important in the development of realistic models for the behavior of composite materials, as discussed in the recent "Panel Discussion on Homogeneization and Damage Localization in Heterogeneous Media" in this year's *Contractors Meeting on Mechanics of Materials* (Dayton, October 1991). Thus, we hope to set the stage for a new project under AFOSR sponsorship. Other potential avenues of research include the consideration of isotropic

composites with nonlinear anisotropic phases (i.e. polycrystalline materials), and studies of texture evolution.

Concerning the development of human resources, we expect that the student supported over the past two years under this project, Mr. Gal de Botton, will complete the requirements for his Ph.D. degreee in the Department of Mechanical Engineering of the University of Pennsylvania sometime towards the second half of the next year.

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Recent Conference Presentations

"The effective properties of anisotropic composites,"
"The effective properties of nonlinear heterogeneous systems,"
"Effective dielectric properties of nonlinear composites" (with G. Li & A.S. Douglas),
28th Annual Meeting of the Society of Engineering Science,
University of Florida, Gainesville, November 6-8, 1991.

"Effective properties in power-law creep."
International Colloquium on Mechanics of Creep Brittle Materials,
Leicester, England, September 2-4, 1991.

"Effective anisotropic properties of creeping composites."
The Third International Symposium on Plasticity and its Current Applications,
Grenoble, France, August 12-16, 1991.

"The effective properties of nonlinear composites."
Second International Conference on Industrial and Applied Mathematics,
Washington, July 8-12, 1991.

Recent Invited Seminars

Department of Mechanical Engineering,
Stanford University, Palo Alto, March 5, 1990.

Department of Materials Science and Mineral Engineering,
University of California, Berkeley, March 6, 1990.

Instituto de Investigaciones en Matematicas Applicadas y en Systemas,
Universidad Nacional Autonoma de Mexico, Mexico City, July 1990.

Department of Mechanical and Aerospace Engineering, *Rutgers University*, December 11, 1991.

Ph.D. Students

Gal de Botton, B.S., M.S. (supported by AFOSR)

Appendices

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THE EFFECTIVE MECHANICAL PROPERTIES OF NONLINEAR ISOTROPIC COMPOSITES

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ABSTRACT

A NEW variational structure is proposed that yields a prescription for the effective energy potentials of nonlinear composites in terms of the corresponding energy potentials for linear composites with the same microstructural distributions. The prescription can be used to obtain bounds and estimates for the effective mechanical properties of nonlinear composites from any bounds and estimates that may be available for the effective properties of linear composites. The main advantages of the procedure are the simplicity of its implementation, the generality of its applications and the strength of its results. The general prescription is applied to three special nonlinear composites: a porous material, a two-phase incompressible composite and a rigidly reinforced material. The results are compared with previously available results for the special case of power-law constitutive behavior.

1. INTRODUCTION

THE PREDICTION of the *effective*, or *overall*, constitutive behavior of composite solid materials is both a practically and theoretically important problem, which draws input from many different disciplines, including materials science, mechanics and mathematics. This paper deals with the theoretical prediction of the effective mechanical properties of heterogeneous materials with *nonlinear* phases that are perfectly bonded to each other, and isotropic. To make sense of the notion of effective properties for the composite, the size of the typical heterogeneity is generally assumed to be small compared to the size of the specimen and the scale of variation of the applied loads. It is further assumed that the effect of the interfaces is negligible, so that the effective properties of the composite are essentially derived from the bulk behavior of the constituent phases.

The corresponding theory for linear composites is fairly well developed, including different approaches to the problem with varying degrees of mathematical sophistication and physical relevance. Thus, exact estimates have been determined for the effective properties of *ad hoc* models of composites; rigorous variational bounds have been given for the properties of *random* composites; and precise definitions and explicit "homogenization" formulae have been proposed for the properties of *periodic* composites. Appropriate, but by no means exhaustive, references dealing with the linear theories are provided by the review articles of WILLIS (1982, 1983) and KOHN (1989), and by the monographs of CHRISTENSEN (1979) and SANCHEZ-PALENCIA

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(1980). By contrast, in the context of nonlinear composites, most of the results are based on *ad hoc* models, such as dilute and self-consistent models. For instance, DUVA and HUTCHINSON (1984) (referred to as DH from here on) gave dilute estimates for the effective properties of a nonlinear porous material, and DUVA (1984) proposed self-consistent estimates of the differential type for a rigidly reinforced, nonlinear material. To the knowledge of the author, the first and only contribution so far dealing with the calculation of rigorous bounds for the effective properties of nonlinear composites is provided by the work of TALBOT and WILLIS (1985) (referred to as TW), which was introduced in WILLIS (1983). These authors extended the well-known Hashin–Shtrikman variational principles to include nonlinear constitutive behavior, and their methods have been applied to a number of examples in different physical contexts. For example, PONTE CASTAÑEDA and WILLIS (1988) (referred to as PCW), and more recently WILLIS (1989), have determined bounds and estimates for the effective properties of nonlinearly viscous (or infinitesimally elastic) materials. Also, PONTE CASTAÑEDA (1989) has provided extensions of the minimum complementary energy and the Talbot–Willis variational principles to finite elasticity, that allowed the calculation of bounds and estimates for the effective properties of a broad class of nonlinearly elastic composites. Additional developments are provided by appropriate extensions of the periodic homogenization formula by MARCELLINI (1978) for problems with convex energy densities, and by MULLER (1987) for finite elasticity (with a non-convex energy density).

In this work, we propose an alternative variational structure that allows the estimation of the effective energy densities of nonlinear composites in terms of the corresponding information for linear composites with the same microstructural distribution. Although the procedure has application to problems in other physical contexts that have a variational characterization (see PONTE CASTAÑEDA, 1990, for an application in conductivity), here we will study the specific application of the theory to composite materials with constitutive behavior characterized by nonlinear viscosity, or by the mathematically analogous theory of nonlinear infinitesimal elasticity. Effective properties are defined in Section 2 by means of the principle of minimum potential energy, and its dual counterpart, the principle of minimum complementary energy. The new structure is developed in Section 3, where it is shown that a general bound (or, alternatively, an estimate) for the linear composite can be translated into a bound (or estimate) for the nonlinear composite. In Section 4, the general procedure is applied to three particular cases of general interest: a porous material, a two-phase incompressible composite and a material reinforced by rigid inclusions. For each of these composites, we give bounds and estimates for their effective properties. In Section 5, the results are specialized further to phases with a power-law type of constitutive behavior, and the results are discussed and compared with previously available results. Finally, in Section 6 some general conclusions are drawn.

2. EFFECTIVE PROPERTIES

We are interested in estimating the effective, or overall, properties of composites with nonlinear material behavior. By a "composite" we mean an idealized material

that corresponds to the limit of a sequence of heterogeneous materials with two distinct length scales: one microscopic l corresponding to the size of the heterogeneity, and one macroscopic L corresponding to the size of the specimen of interest and the scale of variation of the boundary conditions. The effective behavior of the composite is then obtained by considering the limit of the behavior of the sequence of materials as the ratio of scales $\varepsilon = l/L$ tends to zero. The study of the definition and existence of such limit properties is called homogenization theory, and it is an area of current interest in the general mathematics community (KOHN, 1989). However, for the purposes of this work, it will not be necessary to introduce this formalism; we can always rely on our physical intuition to understand the concept of effective properties, and in the analysis that follows, it will suffice to take our composite to be a heterogeneous material with very small, but finite microscale. The effective properties are then understood in an approximate sense.

Consider an n-phase composite occupying a domain Ω, with each phase occupying a sub-domain $\Omega^{(r)}$ ($r = 1, 2, \ldots, n$), and let the stress potential, $U(\boldsymbol{\sigma}, \mathbf{x})$, be expressed in terms of the n homogeneous phase potentials, $U^{(r)}(\boldsymbol{\sigma})$, via

$$U(\boldsymbol{\sigma}, \mathbf{x}) = \sum_{r=1}^{n} \chi^{(r)}(\mathbf{x}) U^{(r)}(\boldsymbol{\sigma}), \tag{2.1}$$

where

$$\chi^{(r)}(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in \Omega^{(r)} \\ 0 & \text{otherwise} \end{cases} \tag{2.2}$$

is the characteristic function of phase r. The phases are assumed to be isotropic, so that the potentials $U^{(r)}(\boldsymbol{\sigma})$ depend on the stress $\boldsymbol{\sigma}$ only through its three principal invariants. Here, we will further assume that the dependence is only through two of the invariants, namely, the mean stress

$$\sigma_m = \tfrac{1}{3} \operatorname{tr} \boldsymbol{\sigma},$$

and the effective stress

$$\sigma_e = \sqrt{\tfrac{3}{2} \mathbf{S} \cdot \mathbf{S}},$$

where $\mathbf{S}$ is the deviator of $\boldsymbol{\sigma}$.

Then, the strain tensor (or the strain-rate tensor, depending on whether we are dealing with nonlinear infinitesimal elasticity or viscosity) $\boldsymbol{\varepsilon}$, which is required to satisfy the compatibility relations

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \tag{2.3}$$

in terms of the displacement (or velocity field) $\mathbf{u}$, is related to the stress $\boldsymbol{\sigma}$, satisfying the equilibrium equations

$$\sigma_{ij,j} = 0, \tag{2.4}$$

via the constitutive relation

$$\varepsilon_{ij} = \frac{\partial U}{\partial \sigma_{ij}}(\boldsymbol{\sigma}, \mathbf{x}). \tag{2.5}$$

The commas in (2.3) and (2.4) denote differentiation, and the summation convention has also been used in (2.4). We assume that the phases are perfectly bonded, so that the displacement (or velocity) is continuous across the interphase boundaries. However, the strains and, therefore, the stresses may be discontinuous across such boundaries, and hence (2.4) must be interpreted in a weak sense, requiring continuity of the traction components of the stress across the interphase boundaries. Also, at least one of the phase potentials is assumed to be non-quadratic in the stress, so that the constitutive response of the material as given by (2.5) is genuinely nonlinear.

The statement of the problem is completed by the selection of an appropriate boundary condition:

$$\sigma_{ij} n_j = \hat{\sigma}_{ij} n_j, \quad \mathbf{x} \in \partial\Omega, \tag{2.6}$$

where $\partial\Omega$ denotes the boundary of the composite, $\mathbf{n}$ is its unit outward normal, and $\hat{\boldsymbol{\sigma}}$ is a given constant symmetric tensor. This uniform constraint condition has some useful properties, discovered by HILL (1963). Let

$$\bar{\boldsymbol{\sigma}} = \int_\Omega \boldsymbol{\sigma}(\mathbf{x})\, dV. \tag{2.7}$$

and

$$\bar{\boldsymbol{\varepsilon}} = \int_\Omega \boldsymbol{\varepsilon}(\mathbf{x})\, dV \tag{2.8}$$

denote the respective averages of the *actual* stress and strain fields in the composite. Here, the scale of Ω has been normalized to have unit volume. Then, we have that

$$\bar{\boldsymbol{\sigma}} = \hat{\boldsymbol{\sigma}}, \tag{2.9}$$

and

$$\bar{\boldsymbol{\varepsilon}} = \frac{1}{2}\int_{\partial\Omega} (\mathbf{u} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{u})\, dS. \tag{2.10}$$

This means that the average stress $\bar{\boldsymbol{\sigma}}$ in the composite is precisely $\hat{\boldsymbol{\sigma}}$, and that the average strain $\bar{\boldsymbol{\varepsilon}}$ can be "measured" in terms of the boundary displacements.

The third property makes use of the principle of minimum complementary energy, which is a variational characterization of the above problem, described by (2.3) to (2.5), and was first introduced by HILL (1956), under the assumption of strict convexity of the nonlinear potential $U(\boldsymbol{\sigma}, \mathbf{x})$. Thus, we define the effective energy for the composite via the relation

$$\tilde{U}(\bar{\boldsymbol{\sigma}}) = \inf_{\boldsymbol{\sigma} \in S(\bar{\boldsymbol{\sigma}})} \bar{U}(\boldsymbol{\sigma}), \tag{2.11}$$

where

$$\bar{U}(\boldsymbol{\sigma}) = \int_\Omega U(\boldsymbol{\sigma}, \mathbf{x})\, dV$$

is the complementary energy functional of the problem at hand, and

$$S(\bar{\boldsymbol{\sigma}}) = \{\boldsymbol{\sigma} | \sigma_{ij,j} = 0 \text{ in } \Omega, \quad \text{and } \sigma_{ij} n_j = \bar{\sigma}_{ij} n_j \text{ on } \partial\Omega\}$$

is the set of statically admissible stresses. Then, if $\tilde{U}(\bar{\boldsymbol{\sigma}})$ is assumed to be differentiable, it can be readily shown that

$$\bar{\varepsilon}_{ij} = \frac{\partial \tilde{U}}{\partial \bar{\sigma}_{ij}}(\bar{\boldsymbol{\sigma}}), \tag{2.12}$$

which yields an effective constitutive relation for the composite in terms of the average variables $\bar{\boldsymbol{\sigma}}$ and $\bar{\boldsymbol{\varepsilon}}$. Given this connection between the effective potential for the composite $\tilde{U}(\bar{\boldsymbol{\sigma}})$ and the effective stress/strain relation, it makes sense to seek information on $\tilde{U}(\bar{\boldsymbol{\sigma}})$. Notice that, under the above assumptions, $\tilde{U}(\bar{\boldsymbol{\sigma}})$ is convex (refer to, for instance, Appendix A of PCW).

A dual formulation can be given by means of the principle of minimum potential energy in terms of the strain potential $W(\boldsymbol{\varepsilon}, \mathbf{x})$, which is obtained from the stress potential $U(\boldsymbol{\sigma}, \mathbf{x})$ via the Legendre (Fenchel) transformation

$$W(\boldsymbol{\varepsilon}, \mathbf{x}) = \sup_{\boldsymbol{\sigma}} \{\boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon} - U(\boldsymbol{\sigma}, \mathbf{x})\}. \tag{2.13}$$

Thus, if we define the effective strain potential of the composite via

$$\tilde{W}(\bar{\boldsymbol{\varepsilon}}) = \inf_{\boldsymbol{\varepsilon} \in K(\bar{\boldsymbol{\varepsilon}})} \bar{W}(\boldsymbol{\varepsilon}), \tag{2.14}$$

where $\bar{W}(\boldsymbol{\varepsilon})$ is the pertinent potential energy functional, and

$$K(\bar{\boldsymbol{\varepsilon}}) = \{\boldsymbol{\varepsilon} | \varepsilon_{ij} = 1/2(u_{i,j} + u_{j,i}) \text{ in } \Omega, \quad \text{and } u_i = \bar{\varepsilon}_{ij} x_j \text{ on } \partial\Omega\}$$

is the set of kinematically admissible strains satisfying a uniform strain boundary condition, we have an effective stress/strain relation for the composite, expressed by

$$\bar{\sigma}_{ij} = \frac{\partial \tilde{W}}{\partial \bar{\varepsilon}_{ij}}(\bar{\boldsymbol{\varepsilon}}), \tag{2.15}$$

where now $\bar{\boldsymbol{\varepsilon}}$, representing the average strain in the composite, is equal to the prescribed uniform strain on the boundary, and $\bar{\boldsymbol{\sigma}}$, representing the average stress, can be "measured" in terms of the traction on the boundary. Notice that $\tilde{W}(\bar{\boldsymbol{\varepsilon}})$ is also convex.

The above development suggests that $\tilde{W}(\bar{\boldsymbol{\varepsilon}})$ and $\tilde{U}(\bar{\boldsymbol{\sigma}})$ might also be related through the Legendre transformation:

$$\tilde{W}(\bar{\boldsymbol{\varepsilon}}) = \sup_{\bar{\boldsymbol{\sigma}}} \{\bar{\boldsymbol{\sigma}} \cdot \bar{\boldsymbol{\varepsilon}} - \tilde{U}(\bar{\boldsymbol{\sigma}})\}. \tag{2.16}$$

However, this is certainly not true for a general heterogeneous material, since the boundary conditions associated with the two formulations are different (uniform traction versus uniform strain), and hence $\tilde{U}(\bar{\boldsymbol{\sigma}})$ and $\tilde{W}(\bar{\boldsymbol{\varepsilon}})$ would correspond to the solutions of different problems. In fact, WILLIS (1990) has shown that replacing the uniform traction boundary condition in $S(\bar{\boldsymbol{\sigma}})$ by the condition that the stresses have

mean value $\bar{\sigma}$ gives exact duality. This implies that, in general, the equality $(=)$ in (2.16) must be replaced by an inequality $(\geq)$. On the other hand, it seems reasonable that in the limit of the microscale tending to zero (the homogenized limit), the response of the composite would be the same for both types of boundary conditions, and thus $\tilde{U}(\bar{\sigma})$ and $\tilde{W}(\bar{\varepsilon})$ are expected to be Legendre duals in that limit. A rigorous proof of this fact, however, would involve a more rigorous definition of the homogenized limit than we have utilized, and is beyond the scope of this work.

At this point, a few remarks are in order. First of all, it should be noted that the three properties (2.9), (2.10) and (2.12) (or the corresponding ones in the dual formulation) hold true for any heterogeneous material, whether it is a "composite" in the sense described above, or not. However, $\tilde{U}(\bar{\sigma})$ is expected to represent the effective properties of some *idealized* homogeneous material, obtained as an appropriately defined mathematical limit of a sequence of heterogeneous materials with vanishingly small microscale. Now, if the microstructure of the composite is deterministic as in a periodic composite, $\tilde{U}(\bar{\sigma})$ can (in principle) be determined uniquely in terms of the solution of a nonlinear boundary problem on a unit cell with periodic boundary conditions (MARCELLINI, 1978). On the other hand, if the microstructure of the composite is random, usually only partial statistical information is available in the form of the volume fractions of the phases, or, less frequently, some higher-order information such as overall isotropy for the composite. It is then not possible to determine the effective properties of any given composite precisely, and it is essential to reinterpret $\tilde{U}(\bar{\sigma})$ as the set of effective energies of a family of composites with some prescribed statistics of the microstructure. In any event, whether the composite is periodic and its effective properties are difficult to find, or random so that its properties are not deterministic, it makes sense to attempt to delimit the effective behavior of composites by specifying *bounds* for $\tilde{U}(\bar{\sigma})$ in terms of some prescribed microstructural information. In some cases, as when the bounds are too far apart to be useful, it may be possible to identify a special solution, called an *estimate*, that in some sense best characterizes the properties of a certain family of composites. In this work, we will only be interested in the case of composites for which the volume fractions of the constituent phases are specified, and that are, in addition, isotropic in an overall sense. In the following sections, we will attempt to specify bounds and estimates for the effective properties of this class of nonlinear composites.

3. THE NEW VARIATIONAL STRUCTURE

In this section, we make use of a linear heterogeneous "comparison" material, with effective properties that can be characterized in terms of bounds and estimates, to obtain corresponding bounds and estimates for the effective properties of a nonlinear composite. To this end, we introduce the following quadratic potential

$$\hat{U}(\boldsymbol{\sigma}, \mathbf{x}) = \sum_{r=1}^{n} \chi^{(r)}(\mathbf{x}) \hat{U}^{(r)}(\boldsymbol{\sigma}) = \frac{1}{6\hat{\mu}(\mathbf{x})} \sigma_e^2 + \frac{1}{2\hat{\kappa}(\mathbf{x})} \sigma_m^2, \tag{3.1}$$

where

$$\hat{\mu}(\mathbf{x}) = \sum_{r=1}^{n} \chi^{(r)}(\mathbf{x}) \hat{\mu}^{(r)} > 0, \quad \text{and} \quad \hat{\kappa}(\mathbf{x}) = \sum_{r=1}^{n} \chi^{(r)}(\mathbf{x}) \hat{\kappa}^{(r)} > 0,$$

with the $\hat{\mu}^{(r)}$ and $\hat{\kappa}^{(r)}$ constant, corresponding to a linear isotropic composite with the same phase distribution (the same characteristic functions) as the nonlinear composite.

Then, if the nonlinearity in the potential of the original composite is stronger than quadratic as the norm of the stress becomes large, it makes sense to define the set of functions

$$V^{(r)}(\hat{\mu}^{(r)}, \hat{\kappa}^{(r)}) = \sup_{\boldsymbol{\sigma}} \{\hat{U}^{(r)}(\boldsymbol{\sigma}) - U^{(r)}(\boldsymbol{\sigma})\}, \tag{3.2}$$

such that

$$V(\hat{\mu}, \hat{\kappa}) = \sum_{r=1}^{n} \chi^{(r)}(\mathbf{x}) V^{(r)}(\hat{\mu}^{(r)}, \hat{\kappa}^{(r)}) = \sup_{\boldsymbol{\sigma}} \{\hat{U}(\boldsymbol{\sigma}, \mathbf{x}) - U(\boldsymbol{\sigma}, \mathbf{x})\}. \tag{3.3}$$

It follows that, for all $\hat{\mu}^{(r)}, \hat{\kappa}^{(r)} > 0$ $(r = 1, \ldots, n)$ and $\boldsymbol{\sigma}$, at each $\mathbf{x} \in \Omega$

$$U(\boldsymbol{\sigma}, \mathbf{x}) \geq \hat{U}(\boldsymbol{\sigma}, \mathbf{x}) - V(\hat{\mu}, \hat{\kappa}),$$

and hence that for all $\hat{\mu}^{(r)}, \hat{\kappa}^{(r)} > 0$ $(r = 1, \ldots, n)$, and for every $\bar{\sigma}$

$$\tilde{U}(\bar{\sigma}) \geq \tilde{\hat{U}}(\bar{\sigma}) - \bar{V}(\hat{\mu}, \hat{\kappa}), \tag{3.4}$$

where

$$\tilde{\hat{U}}(\bar{\sigma}) = \inf_{\sigma \in S(\bar{\sigma})} \hat{U}(\boldsymbol{\sigma}) \tag{3.5}$$

is the effective potential of the linear composite, and

$$\bar{V}(\hat{\mu}, \hat{\kappa}) = \sum_{r=1}^{n} c^{(r)} V^{(r)}(\hat{\mu}^{(r)}, \hat{\kappa}^{(r)}),$$

expressed in terms of the volume fractions of each phase,

$$c^{(r)} = \int_{\Omega} \chi^{(r)}(\mathbf{x}) \, \mathrm{d}V.$$

Thus, if we could compute $\tilde{\hat{U}}(\bar{\sigma})$ for the linear composite in terms of $\hat{\mu}^{(r)}$ and $\hat{\kappa}^{(r)}$, expression (3.4) yields a family of bounds for the effective potential of the nonlinear composite, $\tilde{U}(\bar{\sigma})$, for every choice of the set of parameters $\hat{\mu}^{(r)}, \hat{\kappa}^{(r)} > 0$. This family of bounds can be optimized by considering

$$\tilde{U}_{-}(\bar{\sigma}) = \sup_{\hat{\mu}^{(r)}, \hat{\kappa}^{(r)} > 0} \{\tilde{\hat{U}}(\bar{\sigma}) - \bar{V}(\hat{\mu}, \hat{\kappa})\}. \tag{3.6}$$

Then, evidently,

$$\tilde{U}(\bar{\sigma}) \geq \tilde{U}_{-}(\bar{\sigma}). \tag{3.7}$$

Usually, however, it is not possible to find $\tilde{\hat{U}}(\bar{\sigma})$ explicitly, but instead bounds and estimates may be available for $\tilde{\hat{U}}(\bar{\sigma})$. If we have a lower bound (such as a Hashin–Shtrikman lower bound) $\tilde{\hat{U}}_{-}(\bar{\sigma})$ for $\tilde{\hat{U}}(\bar{\sigma})$, such that

$$\tilde{\tilde{U}}(\tilde{\sigma}) \geqslant \tilde{\tilde{U}}_{-}(\tilde{\sigma}), \tag{3.8}$$

then, replacing $\tilde{\tilde{U}}(\tilde{\sigma})$ by $\tilde{\tilde{U}}_{-}(\tilde{\sigma})$ in (3.6) for $\tilde{U}_{-}(\tilde{\sigma})$, still yields a lower bound for $\tilde{U}(\tilde{\sigma})$; alternatively, an upper bound for $\tilde{\tilde{U}}(\tilde{\sigma})$ is not useful in terms of obtaining an upper bound for $\tilde{U}(\tilde{\sigma})$. On the other hand, if we only have an estimate (such as a self-consistent estimate) $\tilde{\tilde{U}}_{e}(\tilde{\sigma})$ for $\tilde{\tilde{U}}(\tilde{\sigma})$, then

$$\tilde{U}_{e}(\tilde{\sigma}) = \sup_{\hat{\mu}^{(m)}, \hat{\kappa}^{(m)} > 0} \{\tilde{\tilde{U}}_{e}(\tilde{\sigma}) - \bar{V}(\hat{\mu}, \hat{\kappa})\} \tag{3.9}$$

would provide only an estimate for $\tilde{U}(\tilde{\sigma})$.

We note that the prescriptions (3.6) and (3.9) lead to convex expressions for the bounds and estimates of the effective potential of the nonlinear composite, provided that the corresponding bounds and estimates for the linear composite are convex, which is in turn guaranteed assuming that $\hat{\mu}$ and $\hat{\kappa} > 0$. This is a desirable feature, because the effective potential of the composite is known to be convex, and it follows directly from the convexity of $\tilde{\tilde{U}}(\tilde{\sigma})$ in $\tilde{\sigma}$ (the supremum of a family of convex functions is convex).

Although this will not apply to the present work, it is possible that, in other physical contexts, the nonlinearity of the potential for the composite will be weaker than quadratic. In this case, upper bounds $\tilde{U}_{+}(\tilde{\sigma})$ of the form (3.6) and estimates $\tilde{U}_{e}(\tilde{\sigma})$ of the form (3.9) would be obtained for $\tilde{\tilde{U}}(\tilde{\sigma})$, if we replaced the suprema by infima in the definitions (3.2), and the expressions (3.6) for the bound, and (3.9) for the estimate, respectively. In either case, it is important to note that the given prescriptions for the bounds and estimates involve *only* the evaluation of the extreme values of some multidimensional functions, assuming that the corresponding information is available for the linear comparison material.

Before we apply the general procedure developed in this section to some special cases, we show that there is no *duality gap* in our procedure. This is unlike the procedure of TW, which sometimes leads to different bounds and estimates for the effective energy of the nonlinear composite, depending on whether a formulation based on the principle of minimum complementary energy, or on the principle of minimum potential energy is selected.

Using a procedure completely analogous to the above procedure, but starting with the minimum complementary energy formulation, instead of the minimum potential energy formulation, we are led to the following upper bound for the effective potential $\tilde{W}(\tilde{\varepsilon})$:

$$\tilde{W}_{+}(\tilde{\varepsilon}) = \inf_{\hat{\mu}^{(m)}, \hat{\kappa}^{(m)} > 0} \{\tilde{\tilde{W}}(\tilde{\varepsilon}) + \bar{V}(\hat{\mu}, \hat{\kappa})\}, \tag{3.10}$$

where

$$\tilde{\tilde{W}}(\tilde{\varepsilon}) = \inf_{\varepsilon \in K(\tilde{\varepsilon})} \tilde{\tilde{W}}(\varepsilon), \tag{3.11}$$

and

$$V(\hat{\mu}, \hat{\kappa}) = \sup_{\varepsilon} \{W(\varepsilon, \mathbf{x}) - \hat{W}(\varepsilon, \mathbf{x})\}. \tag{3.12}$$

Notice that the use of the same notation for the difference function $V(\hat{\mu}, \hat{\kappa})$ is justified, because, as the following development shows, this present definition of $V(\hat{\mu}, \hat{\kappa})$ is in exact agreement with the prior definition (3.3) of $V(\hat{\mu}, \hat{\kappa})$. Thus,

$$\begin{aligned} V(\hat{\mu}, \hat{\kappa}) &= \sup_{\varepsilon} \left\{ \sup_{\sigma} \{\sigma \cdot \varepsilon - U(\sigma, \mathbf{x})\} - \hat{W}(\varepsilon, \mathbf{x}) \right\} \\ &= \sup_{\sigma} \left\{ \sup_{\varepsilon} \{\sigma \cdot \varepsilon - \hat{W}(\varepsilon, \mathbf{x})\} - U(\sigma, \mathbf{x}) \right\} \\ &= \sup_{\sigma} \{\hat{U}(\sigma, \mathbf{x}) - U(\sigma, \mathbf{x})\}, \end{aligned}$$

where we have used the fact that the order of suprema may be interchanged.

To demonstrate that there is no duality gap in the above procedure, we start with the upper bound for the effective strain potential of the composite

$$\tilde{W}(\tilde{\varepsilon}) \leqslant \tilde{W}_{+}(\tilde{\varepsilon}). \tag{3.13}$$

Then, applying the Legendre transformation to both sides of this inequality, we get (see VAN TIEL, 1984, Section 6.3a)

$$\tilde{U}(\tilde{\sigma}) \geqslant \sup_{\tilde{\varepsilon}} \{\tilde{\sigma} \cdot \tilde{\varepsilon} - \tilde{W}_{+}(\tilde{\varepsilon})\},$$

where we have made use of (2.16), and the fact that both $\tilde{W}(\tilde{\varepsilon})$ and $\tilde{U}(\tilde{\sigma})$ are convex. Next note, from (3.6), that

$$\begin{aligned} \tilde{U}_{-}(\tilde{\sigma}) &= \sup_{\hat{\mu}^{(m)}, \hat{\kappa}^{(m)} > 0} \left\{ \sup_{\tilde{\varepsilon}} \{\tilde{\sigma} \cdot \tilde{\varepsilon} - \tilde{\tilde{W}}(\tilde{\varepsilon})\} - \bar{V}(\hat{\mu}, \hat{\kappa}) \right\} \\ &= \sup_{\tilde{\varepsilon}} \{\tilde{\sigma} \cdot \tilde{\varepsilon} - \inf_{\hat{\mu}^{(m)}, \hat{\kappa}^{(m)} > 0} \{\tilde{\tilde{W}}(\tilde{\varepsilon}) + \bar{V}(\hat{\mu}, \hat{\kappa})\}\} \\ &= \sup_{\tilde{\varepsilon}} \{\tilde{\sigma} \cdot \tilde{\varepsilon} - \tilde{W}_{+}(\tilde{\varepsilon})\}, \end{aligned}$$

where we have made use of some of the properties of infima and suprema, and assumed that duality in the form of (2.16) also applies for the linear composite. Thus, we conclude that the two bounds obtained above for the effective energy of the nonlinear composite (one arising from the potential energy principle, and the other from the complementary energy principle) are equivalent, and hence there is no duality gap. More generally, we can apply the same ideas to the specialized versions of the bounds, and again we would obtain exact duality of the bounds, even for a general (not necessarily a "composite") heterogeneous material. For instance, if we replace $\tilde{\tilde{U}}(\tilde{\sigma})$ by its linear HS lower bound, and $\tilde{\tilde{W}}(\tilde{\varepsilon})$ by its linear HS upper bound, then the exact duality of the linear HS bounds translates into exact duality for the nonlinear bounds. A more formal development of these ideas, with some additional results, is given in PONTE CASTAÑEDA (1990).

4. APPLICATIONS

4.1. *The porous composite*

In this sub-section, we apply the general procedure of Section 3 to a two-phase isotropic composite with one vacuous phase. We take the other phase to be incompressible, with potential

$$U^{(1)}(\boldsymbol{\sigma}) = f(\sigma_e), \tag{4.1}$$

where f is a strictly convex function of its argument with stronger than quadratic growth as the argument becomes large. Then, the stress/strain relation for this phase is given by

$$\varepsilon_{ij} = \frac{3}{2}\frac{f'(\sigma_e)}{\sigma_e} S_{ij}, \tag{4.2}$$

so the the effective strain, $\varepsilon_e = \sqrt{\frac{2}{3}\mathbf{e}\cdot\mathbf{e}}$ ($\mathbf{e}$ is the deviator of $\boldsymbol{\varepsilon}$), is related to the effective stress via

$$\varepsilon_e = \tfrac{1}{2} f'(\sigma_e).$$

On the other hand, the potential of the vacuous phase is given by

$$U^{(2)}(\boldsymbol{\sigma}) = \delta_0(\sigma_e) + \delta_0(\sigma_m), \tag{4.3}$$

where

$$\delta_0(x) = \begin{cases} 0 & \text{if } x = 0 \\ \infty & \text{otherwise.} \end{cases}$$

Now, if we let

$$\hat{U}^{(1)}(\boldsymbol{\sigma}) = \frac{1}{6\hat{\mu}^{(1)}}\sigma_e^2 \tag{4.4}$$

and

$$\hat{U}^{(2)}(\boldsymbol{\sigma}) = \delta_0(\sigma_e) + \delta_0(\sigma_m), \tag{4.5}$$

we have that

$$V^{(1)}(\hat{\mu}^{(1)}) = \tfrac{1}{2}\sigma f'(\sigma) - f(\sigma), \tag{4.6}$$

where σ is some function of $\hat{\mu}^{(1)}$ determined by the optimization problem (3.2), and also that

$$V^{(2)}(\hat{\mu}^{(2)}) = 0. \tag{4.7}$$

It follows from (3.6) that

$$\tilde{U}_-(\bar{\boldsymbol{\sigma}}) = \sup_{\hat{\mu}^{(1)} > 0} \{\tilde{U}(\bar{\boldsymbol{\sigma}}) - c^{(1)} V^{(1)}(\hat{\mu}^{(1)})\}, \tag{4.8}$$

provides a general lower bound for $\tilde{U}(\boldsymbol{\sigma})$ given $\tilde{U}(\boldsymbol{\sigma})$. As discussed previously, in general, we do not know $\tilde{U}(\boldsymbol{\sigma})$ precisely. Here, we will make use of the Voigt and lower Hashin–Shtrikman (HS) bounds for $\tilde{U}(\boldsymbol{\sigma})$ to obtain corresponding bounds for $\tilde{U}(\boldsymbol{\sigma})$. Additionally, we will provide a self-consistent (SC) estimate for $\tilde{U}(\boldsymbol{\sigma})$ in terms of the well-known SC estimate for $\tilde{U}(\boldsymbol{\sigma})$.

4.1.1. *Voigt bound.* For the linear composite with one vacuous phase, it is known that

$$\tilde{U}(\bar{\boldsymbol{\sigma}}) \geqslant \frac{1}{6\hat{\mu}_V}\bar{\sigma}_e^2,$$

where $\hat{\mu}_V = c^{(1)}\hat{\mu}^{(1)}$. Therefore, from relation (4.8), we have that

$$\begin{aligned}\tilde{U}(\bar{\boldsymbol{\sigma}}) &\geqslant c^{(1)} \sup_{\hat{\mu}^{(1)} > 0} \left\{\frac{1}{6\hat{\mu}^{(1)}} s^2 - V^{(1)}(\hat{\mu}^{(1)})\right\} \\ &= c^{(1)} f(s),\end{aligned} \tag{4.9}$$

where

$$s = \bar{\sigma}_e / c^{(1)}, \tag{4.10}$$

and where we have made use of the result of the Appendix, and hence we must further assume that $F(x) = f(s)$ is a convex function of $x = s^2 > 0$. The result expressed by (4.9) and (4.10) is precisely the nonlinear Voigt bound, which could alternatively be obtained from the principle of minimum potential energy by assuming a uniform strain field throughout the composite, and dualizing the result. Thus, at least in this simple case, the new procedure reproduces the "right" result, exactly.

4.1.2. *Hashin–Shtrikman bound.* For linear isotropic composites, HASHIN and SHTRIKMAN (1962) found upper and lower bounds that are tighter than the Voigt–Reuss bounds. The lower bound for our particular example specializes to

$$\tilde{U}(\bar{\boldsymbol{\sigma}}) \geqslant \frac{1}{6\hat{\mu}_{HS}}\bar{\sigma}_e^2 + \frac{1}{2\hat{\kappa}_{HS}}\bar{\sigma}_m^2,$$

where

$$\hat{\mu}_{HS} = \frac{c^{(1)}\hat{\mu}^{(1)}}{1 + \frac{2}{3}c^{(2)}} \quad \text{and} \quad \hat{\kappa}_{HS} = \frac{4}{3}\frac{c^{(1)}}{c^{(2)}}\hat{\mu}^{(1)}.$$

Then, a bound of the form (4.9) applies, where now

$$s = \frac{1}{c^{(1)}}\sqrt{(1 + \tfrac{2}{3}c^{(2)})\bar{\sigma}_e^2 + \tfrac{9}{5}c^{(2)}\bar{\sigma}_m^2}, \tag{4.11}$$

and again we have made use of the result of the Appendix. Because this bound was derived from the linear HS bound, we refer to it as the nonlinear HS bound. Note that, unlike the Voigt bound, it predicts overall compressibility for the composite, as expected physically.

4.1.3. *Self-consistent estimate.* For linear isotropic composites, BUDIANSKY (1965) and HILL (1965) provided the so-called SC estimates for the effective moduli. In this particular case, these estimates specialize to

$$\tilde{U}(\bar{\boldsymbol{\sigma}}) \approx \frac{1}{6\hat{\mu}_{SC}}\bar{\sigma}_e^2 + \frac{1}{2\hat{\kappa}_{SC}}\bar{\sigma}_m^2,$$

where

$$\hat{\mu}_{SC} = \frac{(1-2c^{(2)})}{(1-c^{(2)}/3)}\hat{\mu}^{(1)}, \quad \text{and} \quad \hat{\kappa}_{SC} = \frac{4}{3}\frac{c^{(1)}}{c^{(2)}}\frac{(1-2c^{(2)})}{(1-c^{(2)}/3)}\hat{\mu}^{(1)}.$$

Then

$$\tilde{U}(\bar{\boldsymbol{\sigma}}) \approx c^{(1)} f(s),$$

where now

$$s = \sqrt{\frac{1}{c^{(1)}}\left(\frac{1-c^{(2)}/3}{1-2c^{(2)}}\right)\left(\bar{\sigma}_e^2 + \frac{9}{4}\frac{c^{(2)}}{c^{(1)}}\bar{\sigma}_m^2\right)}, \tag{4.12}$$

and once again we have made use of the Appendix. We refer to this result as the nonlinear SC estimate.

4.2. *The two-phase incompressible composite*

In this sub-section, we consider a general two-phase composite with isotropic, incompressible and nonlinear phases such that

$$U^{(r)}(\boldsymbol{\sigma}) = f^{(r)}(\sigma_e), \quad (r = 1, 2), \tag{4.13}$$

where the $f^{(r)}$ satisfy the same convexity conditions of the previous section. Then if we let

$$\hat{U}^{(r)}(\boldsymbol{\sigma}) = \frac{1}{6\hat{\mu}^{(r)}}\sigma_e^2, \tag{4.14}$$

we have that

$$V^{(r)}(\hat{\mu}^{(r)}) = \tfrac{1}{2}\sigma^{(r)} f^{(r)\prime}(\sigma^{(r)}) - f^{(r)}(\sigma^{(r)}), \tag{4.15}$$

where the $\sigma^{(r)}$ are some functions of the $\hat{\mu}^{(r)}$ that are determined by the solution of the optimization problem (3.2). It follows that

$$\tilde{U}(\bar{\boldsymbol{\sigma}}) = \sup_{\hat{\mu}^{(1)},\hat{\mu}^{(2)}>0}\{\tilde{\hat{U}}(\bar{\boldsymbol{\sigma}}) - c^{(1)}V^{(1)}(\hat{\mu}^{(1)}) - c^{(2)}V^{(2)}(\hat{\mu}^{(2)})\} \tag{4.16}$$

is a general lower bound for $\tilde{U}(\bar{\boldsymbol{\sigma}})$, given $\tilde{\hat{U}}(\bar{\boldsymbol{\sigma}})$. As in the previous sub-section, we use this result to obtain the Voigt bound, a HS bound and a SC estimate for the effective potential of the nonlinear composite in terms of the corresponding results for the linear composite.

4.2.1. *Voigt bound.* In this case, it is more convenient to consider the dual formulation. For the linear composite, we have that

$$\tilde{\hat{W}}(\bar{\boldsymbol{\varepsilon}}) \leqslant \tfrac{1}{2}\hat{\mu}_1\bar{\varepsilon}_e^2 + \delta_0(\bar{\varepsilon}_m), \tag{4.17}$$

where $\hat{\mu}_1 = c^{(1)}\hat{\mu}^{(1)} + c^{(2)}\hat{\mu}^{(2)}$. Then, it can easily be shown that

$$\tilde{W}(\bar{\boldsymbol{\varepsilon}}) \leqslant c^{(1)}W^{(1)}(\bar{\boldsymbol{\varepsilon}}) + c^{(2)}W^{(2)}(\bar{\boldsymbol{\varepsilon}}), \tag{4.18}$$

where we have made use of the appropriate specialization of (3.10), and a dual version of the result of the Appendix. As in the previous example, this result agrees exactly with the nonlinear Voigt bound.

4.2.2. *Hashin–Shtrikman bounds.* For the linear composite, we have

$$\tilde{\hat{U}}(\bar{\boldsymbol{\sigma}}) \geqslant \frac{1}{6\hat{\mu}_{HS}}\bar{\sigma}_e^2, \tag{4.19}$$

where

$$\hat{\mu}_{HS} = \frac{c^{(1)}\hat{\mu}^{(1)}(6\hat{\mu}^{(2)}+9\hat{\mu}) + c^{(2)}\hat{\mu}^{(2)}(6\hat{\mu}^{(1)}+9\hat{\mu})}{c^{(1)}(6\hat{\mu}^{(2)}+9\hat{\mu}) + c^{(2)}(6\hat{\mu}^{(1)}+9\hat{\mu})}$$

and $\hat{\mu} = \max\{\hat{\mu}^{(1)}, \hat{\mu}^{(2)}\} = \hat{\mu}^{(1)}$ (by assumption). Then, it follows that

$$\tilde{U}(\bar{\boldsymbol{\sigma}}) \geqslant \sup_{\hat{\mu}^{(1)},\hat{\mu}^{(2)}>0}\left\{\frac{1}{6\hat{\mu}_{HS}}\bar{\sigma}_e^2 - c^{(1)}V^{(1)}(\hat{\mu}^{(1)}) - c^{(2)}V^{(2)}(\hat{\mu}^{(2)})\right\} \tag{4.20}$$

but, unfortunately, no further simplification is possible in general, due to the coupling of $\hat{\mu}^{(1)}$ and $\hat{\mu}^{(2)}$ in the term involving $\hat{\mu}_{HS}$. Later we will apply this result to a special case where $f^{(1)}, f^{(2)}$ have a simple power-law form.

4.2.3. *Self-consistent estimates.* The results for the SC estimates have the same forms (4.19) and (4.20) as the HS bounds, but with $\hat{\mu}_{HS}$ replaced by $\hat{\mu}_{SC}$, given by the positive root of the expression

$$3\hat{\mu}_{SC}^2 + \{(2-5c^{(1)})\hat{\mu}^{(1)} + (2-5c^{(2)})\hat{\mu}^{(2)}\}\hat{\mu}_{SC} - 2\hat{\mu}^{(1)}\hat{\mu}^{(2)} = 0. \tag{4.21}$$

As was the case for the HS bound, these results cannot be simplified further without specifying in more detail the constitutive behavior of the phase materials.

4.3. *The composite reinforced by rigid inclusions*

This is a special case of the previous material with $f^{(1)}(\sigma_e) = 0$, corresponding to the case where phase #1 is rigid. Then, the choice $\hat{U}^{(1)}(\boldsymbol{\sigma}) = 0$ leads to $V^{(1)}(\hat{\mu}^{(1)}) = 0$, and we have the following general lower bound for the effective energy of the composite:

$$\tilde{U}(\bar{\boldsymbol{\sigma}}) \geqslant \sup_{\hat{\mu}^{(2)}>0}\{\tilde{\hat{U}}(\bar{\boldsymbol{\sigma}}) - c^{(2)}V^{(2)}(\mu^{(2)})\}. \tag{4.22}$$

In this case, however, it is clear that the lower bounds are trivial, and we will only be able to give estimates for the effective energy of the composite. We will consider the standard SC estimate, as well as a differential self-consistent (DSC) estimate.

4.3.1. *Self-consistent estimate.* For the linear composite, it is known that

$$\tilde{\hat{U}}(\bar{\boldsymbol{\sigma}}) \approx \frac{1}{6\hat{\mu}_{SC}}\bar{\sigma}_e^2, \tag{4.23}$$

where

$$\hat{\mu}_{SC} = \frac{\hat{\mu}^{(2)}}{(1-\frac{5}{3}c^{(1)})} \quad (c^{(1)} < \tfrac{3}{5}).$$

Using this result in the general expression (4.22) induces a nonlinear SC estimate for the effective energy

$$\tilde{U}(\bar{\sigma}) \approx c^{(2)} f^{(2)}(s), \tag{4.24}$$

where

$$s = \left(\frac{1-\frac{5}{3}c^{(1)}}{c^{(2)}}\right)^{1/2} \bar{\sigma}_e, \tag{4.25}$$

and once again we have made use of the result in the Appendix.

4.3.2. *Differential self-consistent estimate.* The DSC estimates of BOUCHER (1974), and MCLAUGHLIN (1977) for linear elastic composites specialize in this case to expression (4.23), with $\hat{\mu}_{SC}$ replaced by

$$\hat{\mu}_{DSC} = \frac{\hat{\mu}^{(2)}}{(c^{(2)})^{5/2}}. \tag{4.26}$$

This induces a nonlinear DSC estimate via (4.22) that reduces to an expression similar to (4.24), where now

$$s = (c^{(2)})^{3/4}\bar{\sigma}_e. \tag{4.27}$$

5. RESULTS FOR POWER-LAW MATERIALS

In this section, we specialize further the calculations of the previous section by taking the constitutive behavior of the phases to be governed by a power law relation

$$f^{(r)}(\sigma_e) = \frac{1}{3}\left(\frac{2}{n+1}\right)\frac{1}{(2\mu^{(r)})^n}\sigma_e^{n+1}. \tag{5.1}$$

This class of functions clearly satisfies all the assumptions invoked in Sections 3 and 4, including the convexity assumption of the Appendix. Additionally, we compare the new results for the bounds and estimates with previously available results.

5.1. *The porous material*

In this case, our results for the Voigt bound, the HS bound and the SC estimate all take the simple form of expression (4.9), with s given by (4.10), (4.11) and (4.12), respectively. We compare these results with the results of PCW for the same material. We note, however, that WILLIS (1989) has given a HS bound and several SC estimates for the general case considered in Section 4.1, but the form of his results is more complicated than our new results. For this reason, and because we expect the comparison of the power-law results to be fairly representative, we do not make a more general comparison between our new results and those of WILLIS (1989). There are two domains of particular interest: the low-triaxiality range corresponding to $\omega = |\bar{\sigma}_m/\bar{\sigma}_e| \ll 1$, and the high-triaxiality range corresponding to $\omega \gg 1$.

For low-triaxiality, all the results for the bounds and estimates take the form

$$\tilde{U}(\bar{\sigma}) \approx \frac{1}{3}\left(\frac{2}{n+1}\right)\frac{1}{(2\tilde{\mu})^n}[1+b(c^{(2)},n)\omega^2]\bar{\sigma}_e^{n+1}, \tag{5.2}$$

and it suffices to compare all the corresponding values of the low-triaxiality modulus $\tilde{\mu}$, as functions of $\mu^{(1)}$, $c^{(2)}$ and n. We give the results below, where we identify the source of the results in parentheses:

(Voigt) $$\frac{\tilde{\mu}_1}{\mu^{(1)}} = c^{(1)},$$

(HS) $$\frac{\tilde{\mu}_{HS}}{\mu^{(1)}} = \frac{c^{(1)}}{(1+\frac{5}{3}c^{(2)})^{(n+1)/2n}},$$

(SC) $$\frac{\tilde{\mu}_{SC}}{\mu^{(1)}} = (c^{(1)})^{(n-1)/2n}\left(\frac{1-2c^{(2)}}{1-\frac{5}{3}c^{(2)}}\right)^{(n+1)/2n},$$

(HS : PCW) $$\frac{\tilde{\mu}_{HS}}{\mu^{(1)}} = c^{(1)}\frac{\left(1+\dfrac{n+1}{3n}c^{(2)}\right)}{(1+\frac{5}{3}c^{(2)})^{(n+1)/n}},$$

(SC : PCW) $$\frac{\tilde{\mu}_{SC}}{\mu^{(1)}} = (c^{(1)})^{(n-1)/n}\left(\frac{1-2c^{(2)}}{1-\frac{5}{3}c^{(2)}}\right)^{1/n}\left[1-\left(\frac{n-1}{3n}\right)\frac{c^{(2)}}{c^{(1)}}\left(\frac{1+\frac{1}{2}c^{(2)}}{1-\frac{5}{3}c^{(2)}}\right)\right],$$

(D) $$\frac{\tilde{\mu}_D}{\mu^{(1)}} = 1-\frac{n+1}{n}(f^* + \tfrac{1}{2}kx^{*2})c^{(2)}. \tag{5.3}$$

The last expression corresponds to a dilute concentration of voids, and was carried out by DH, who made use of the results of BUDIANSKY *et al.* (1982). The values of f^*, k and x^* as functions of n are taken from that reference.

In the linear limit ($n = 1$), all the above results agree with the well-known results from the linear theory. For small volume fractions of the vacuous phase ($c^{(2)}$), the HS, SC and dilute results agree to first order, and for larger volume fractions the dilute result lies between the HS upper bound and the SC estimate. Figure 1 gives the results for the bounds and estimates of the effective low-triaxiality moduli $\tilde{\mu}$ as functions of $c^{(2)}$, for two distinct nonlinear cases ($n = 3$ and 10). These results are similar to the linear results, with the HS upper bounds lying below the Voigt bound, and the SC estimates close to the dilute results for moderate volume fractions, but approaching the same percolation limit ($c^{(2)} = 1/2$) as the corresponding linear results. Comparing the *new* bound and estimate with those of PCW, we observe that the new results lie slightly below the old results. For the bound, this has the implication that the new bound is an *improvement* on the old bound, at least in the low-triaxiality range.

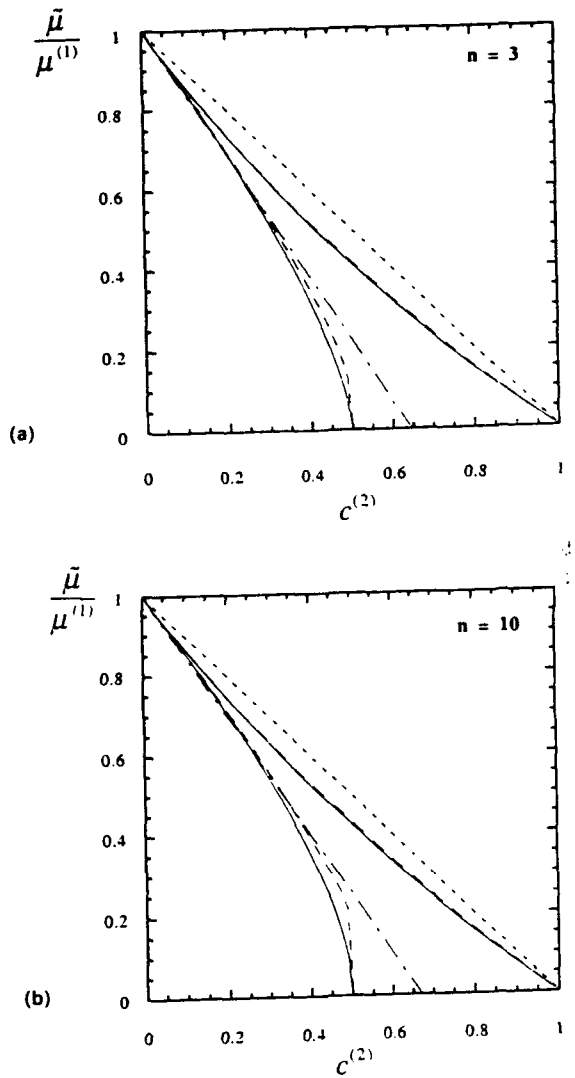


FIG. 1. Bounds and estimates for the low-triaxiality modulus of the porous material as functions of the volume fraction of voids. The short-dash line corresponds to the Voigt bound; the continuous lines correspond to the *new* HS upper bound and SC estimate; the long-dash lines correspond to the HS upper bound and SC estimate of PCW; and the long/short-dash line corresponds to the dilute calculation of DH. Case (a) is for $n = 3$, and (b) for $n = 10$.

For high triaxiality, all the results for the bounds and estimates take the form

$$\tilde{U}(\bar{\boldsymbol{\sigma}}) \approx \frac{1}{3}\left(\frac{2}{n+1}\right)\frac{1}{(2\tilde{\gamma})^n}\left|\frac{3}{2}\bar{\sigma}_m\right|^{n+1}. \tag{5.4}$$

and thus we express all the results in terms of the high-triaxiality moduli $\tilde{\gamma}$, so that

(Voigt) $$\frac{\tilde{\gamma}_V}{\mu^{(1)}} = \infty,$$

(HS) $$\frac{\tilde{\gamma}_{HS}}{\mu^{(1)}} = \frac{c^{(1)}}{(c^{(2)})^{(n+1)/2n}},$$

(SC) $$\frac{\tilde{\gamma}_{SC}}{\mu^{(1)}} = \frac{c^{(1)}}{(c^{(2)})^{(n+1)/2n}}\left(\frac{1-2c^{(2)}}{1-\frac{1}{3}c^{(2)}}\right)^{(n+1)/2n},$$

(HS : PCW) $$\frac{\tilde{\gamma}_{HS}}{\mu^{(1)}} = \frac{c^{(1)}}{(c^{(2)})^{(n+1)/2n}}\left[\left(\frac{2n}{n-1}\right)\left(1+\frac{n+1}{3n}c^{(2)}\right)\right]^{(n-1)/2n},$$

(SC : PCW) $$\frac{\tilde{\gamma}_{SC}}{\mu^{(1)}} = \frac{c^{(1)}}{(c^{(2)})^{(n+1)/2n}}\left(\frac{1-2c^{(2)}}{1-\frac{1}{3}c^{(2)}}\right)^{1/n} \times\left\{\left(\frac{2n}{n-1}\right)c^{(1)}\left[1-\left(\frac{n-1}{3n}\right)\frac{c^{(2)}}{c^{(1)}}\left(\frac{1+\frac{1}{2}c^{(2)}}{1-\frac{1}{3}c^{(2)}}\right)\right]\right\}^{(n-1)/2n},$$

(D) $$\frac{\tilde{\gamma}_D}{\mu^{(1)}} = \frac{n}{(c^{(2)})^{1/n}}, \quad (n < \infty). \tag{5.5}$$

In the linear limit ($n = 1$), the expressions for the HS bounds and SC estimates reduce to the linear results, and, additionally, they agree in the dilute limit ($\tilde{\gamma} \sim (c^{(2)})^{-1}$). Figure 2 depicts plots of the HS bounds and SC and dilute estimates of the high-triaxiality modulus appropriately normalized ($(c^{(2)})^{(n+1)/2n}\tilde{\gamma}$) versus the volume fraction of the void phase ($c^{(2)}$) for two values of the nonlinearity parameter ($n = 3$ and 10). By comparison with the results of PCW, we find that the new results provide a significant improvement over the old results, since the new bounds lie significantly below the old bounds, and are hence tighter. Similarly, the old SC estimate violates the new bound, and must be discarded in favor of the new SC estimate. The dilute result of DH lies below the HS bound for small volume fractions of the void phase ($\tilde{\gamma}_D \sim (c^{(2)})^{-(1/n)}$, whereas $\tilde{\gamma}_{HS} \sim (c^{(2)})^{-(n+1)/2n}$), but it is clear from the plots that the range of validity of the dilute result is severely limited for larger values of n. On the other hand, we should emphasize that the *new* expressions for $\tilde{\gamma}$ are not valid for small values of $c^{(2)}$. This is because the original expression for $\tilde{U}(\bar{\boldsymbol{\sigma}})$, from which they derive, is indeterminate in the limit as $\omega \to \infty$ and $c^{(2)} \to 0$. A better comparison of the new results with the dilute results is accomplished by making use of the original expression for $\tilde{U}(\bar{\boldsymbol{\sigma}})$ in the form

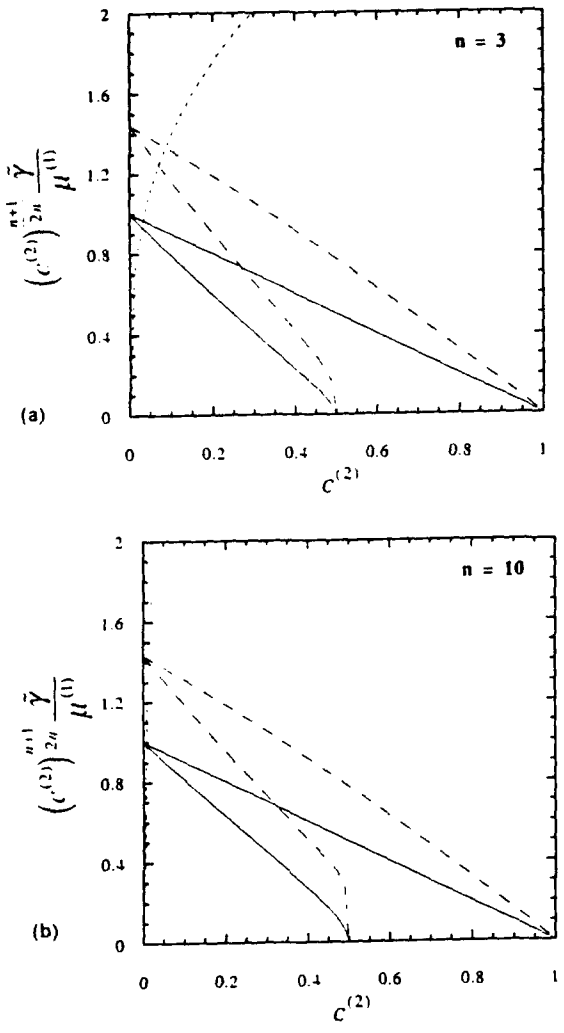


FIG. 2. Bounds and estimates for the high-triaxiality modulus of the porous material as functions of the volume fraction of voids. The continuous lines correspond to the *new* HS upper bound and SC estimate; the long-dash lines correspond to the HS upper bound and SC estimate of PCW; and the short-dash line corresponds to the dilute calculation of DH. Case (a) is for $n = 3$, and (b) for $n = 10$.

$$\tilde{U}(\bar{\sigma}) = \frac{1}{3}\left(\frac{2}{n+1}\right)\frac{1}{(2\tilde{\mu}^*)^n}\bar{\sigma}_e^{n+1}, \tag{5.6}$$

where now $\tilde{\mu}^*$ is a function of $c^{(2)}$, n and ω. Thus, we have that

$$\frac{\tilde{\mu}^*_{HS}}{\mu^{(1)}} = \frac{c^{(1)}}{[1+c^{(2)}(\frac{2}{3}+\frac{9}{4}\omega^2)]^{(n+1)/2n}}, \tag{5.7}$$

and

$$\frac{\tilde{\mu}^*_{D}}{\mu^{(1)}} = \frac{1}{[1+c^{(2)}(n+1)f(\omega,n)]^{1/n}}, \tag{5.8}$$

where $f(\omega, n)$ is taken from the work DH.

Written in this form, it is clear that both expressions for the effective energy of the composite are indeterminate in the limit as $\omega \to \infty$ and $c^{(2)} \to 0$. The first expression, however, is general, whereas the second assumes that $c^{(2)} \ll 1$, and its range of validity is not known. Figure 3 shows a comparison of these two results as functions of ω for two values of n (3 and 10) and three values of $c^{(2)}$ (0.1, 0.01 and 0.001). It is apparent that the dilute approximation of DH is acceptable (although this is not a proof that it is correct) for very small values of $c^{(2)}$ in the sense that it does not violate the new bound, but unacceptable for values of $c^{(2)}$ in the order of 1 to 10%, or larger, depending on the specific value of n. On the other hand, it appears that the new bound could conceivably be subject to improvement for very small values of $c^{(2)}$.

5.2. *The two-phase incompressible composite*

In the case when $f^{(1)}$ and $f^{(2)}$ have the same form of (5.1), but with different moduli $\mu^{(1)}$ and $\mu^{(2)}$, respectively, all the results of Section 4.2 take the form

$$\tilde{U}(\bar{\sigma}) = \frac{1}{3}\left(\frac{2}{n+1}\right)\frac{1}{(2\tilde{\mu})^n}\bar{\sigma}_e^{n+1}, \tag{5.9}$$

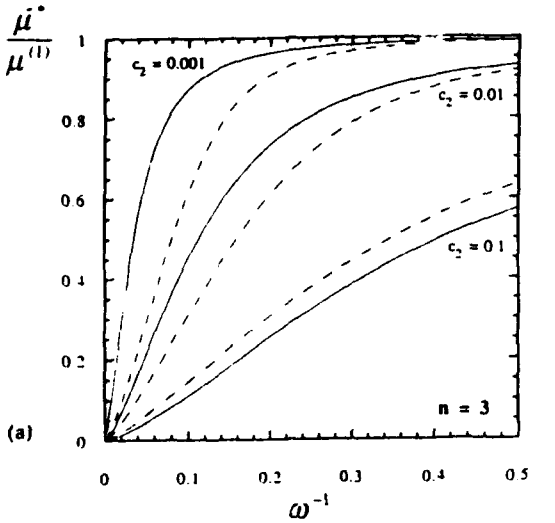
and it suffices to compare the bounds and estimates for the effective modulus $\tilde{\mu}$. The Voigt bound is obtained from (4.18), and is given by

$$\frac{\tilde{\mu}_V}{\mu^{(1)}} = c^{(1)} + c^{(2)}\frac{\mu^{(2)}}{\mu^{(1)}}. \tag{5.10}$$

The new HS bound is obtained by solving the optimization problem given in (4.20); the result is

$$\frac{\tilde{\mu}_{HS}}{\mu^{(1)}} = \left[c^{(1)}S_{(1)}^{n+1} + c^{(2)}\left(\frac{\mu^{(1)}}{\mu^{(2)}}\right)^n S_{(2)}^{n+1}\right]^{-(1/n)}, \tag{5.11}$$

where $S_{(1)}$ and $S_{(2)}$ satisfy the relations



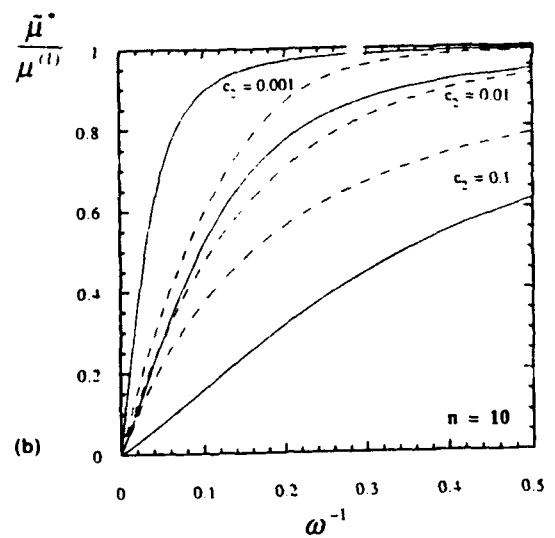


FIG. 3. Bounds and dilute estimates for the high-triaxiality modulus of the porous material as functions of the triaxiality for three different volume fractions of voids. The continuous line corresponds to the *new* HS upper bound, and the dash line corresponds to the dilute calculation of DH. Case (a) is for $n = 3$, and (b) for $n = 10$.

$$[5-(2+3c^{(2)})S_{(2)}]S_{(1)}^{n-1} = 3c^{(1)}\left(\frac{\mu^{(1)}}{\mu^{(2)}}\right)^n S_{(2)}^n$$

and

$$15c^{(1)}S_{(1)}^2 = 6c^{(1)}S_{(2)} + \frac{1}{c^{(1)}}[3+c^{(2)}(2-5S_{(2)})][5-(2+3c^{(2)})S_{(2)}], \qquad (5.12)$$

subject to the restriction that

$$\left(\frac{S_{(2)}}{S_{(1)}}\right) > \left(\frac{\mu^{(1)}}{\mu^{(2)}}\right)^{n/(n-1)}.$$

The SC estimate $\tilde{U}_{SC}$ is obtained from (4.20), and the result has the same form as (5.11), but now $S_{(1)}$ and $S_{(2)}$ satisfy

$$c^{(1)}S_{(1)}^2 + c^{(2)}\frac{\tilde{\mu}^{(1)}}{\tilde{\mu}^{(2)}}S_{(2)}^2 = \frac{\tilde{\mu}^{(1)}}{\hat{\mu}_{SC}}$$

and

$$\left[3\frac{\hat{\mu}_{SC}}{\tilde{\mu}^{(1)}} + (2-5c^{(1)})\right]\frac{\tilde{\mu}^{(2)}}{\tilde{\mu}^{(1)}} = c^{(2)}\left[6\frac{\hat{\mu}_{SC}}{\tilde{\mu}^{(1)}} + (2-5c^{(1)}) + (2-5c^{(2)})\frac{\tilde{\mu}^{(2)}}{\tilde{\mu}^{(1)}}\right]\frac{\hat{\mu}_{SC}}{\tilde{\mu}^{(1)}}S_{(2)}^2, \qquad (5.13)$$

where

$$\frac{\tilde{\mu}^{(1)}}{\tilde{\mu}^{(2)}} = \left(\frac{\mu^{(1)}}{\mu^{(2)}}\right)^n\left(\frac{S_{(2)}}{S_{(1)}}\right)^{n-1}$$

and $\hat{\mu}_{SC}$ is the positive root of (4.21). Additionally, for this problem we have a non-trivial Reuss lower bound (if $\mu^{(2)} > 0$) given by

$$\frac{\tilde{\mu}_R}{\mu^{(1)}} = \left[c^{(1)} + c^{(2)}\left(\frac{\mu^{(1)}}{\mu^{(2)}}\right)^n\right]^{-1/n}. \qquad (5.14)$$

In the linear limit ($n = 1$), the above results reduce to the well-known results from the linear theory and additionally there is a HS lower bound for $\tilde{\mu}$. The HS bounds are nested within the Reuss/Voigt bounds, and the SC estimate lies within the HS bounds, agreeing with the HS upper bound for small volume fractions of phase #2 ($c^{(2)}$), and with the HS lower bound for small volume fractions of phase #1 ($c^{(1)} = 1-c^{(2)}$). As the ratio $\mu^{(2)}/\mu^{(1)}$ becomes smaller, the bounds spread out until eventually (when $\mu^{(2)}/\mu^{(1)} = 0$) the lower bounds become trivial and the SC estimate

reaches a percolation limit at $c^{(2)} = 3/5$. Figure 4 depicts the corresponding results (except the HS lower bound) for the two usual values of n (3 and 10). By comparison with the linear results (not shown), they roughly show the same trends. However, the following comparisons can be established: the Voigt bound is unaffected; the SC estimate is shifted up for low values of $c^{(2)}$, and down slightly for low values of $c^{(1)}$, with larger values of n; and the Reuss bound is shifted down significantly with larger values of n to the point of being nearly without practical value for large n. Extrapolating from the linear theory, and according to the new results, we expect the SC estimates to be good at predicting the effective moduli of the composite, if $\mu^{(2)}/\mu^{(1)}$ is not too small.

5.3. *The composite reinforced by rigid inclusions*

In this case, as in the previous one, all the results take the form of relation (5.1), where now we no longer have bounds for $\tilde{\mu}$, but only estimates, including a SC as well as a DSC estimate.

The SC estimate is obtained from expressions (4.24) and (4.25) and is given by

$$\frac{\tilde{\mu}_{SC}}{\mu^{(2)}} = \frac{(c^{(2)})^{(n-1)/2n}}{(1-\tfrac{5}{3}c^{(1)})^{(n+1)/2n}}. \tag{5.15}$$

The corresponding result from PCW is given by

$$\frac{\tilde{\mu}_{SC}}{\mu^{(2)}} = \frac{\left(1-\dfrac{7n-3}{4n}c^{(1)}\right)}{(1-\tfrac{5}{3}c^{(1)})(c^{(2)})^{1/n}}. \tag{5.16}$$

Our DSC estimate is obtained from the corresponding linear estimate via expression (4.24), together with (4.27) and is given by

$$\frac{\tilde{\mu}_{DSC}}{\mu^{(2)}} = \frac{1}{(c^{(2)})^{(3n+7)/4n}}. \tag{5.17}$$

DUVA (1984) has also provided a DSC estimate, based on computations for an isolated rigid inclusion in an infinite power-law viscous matrix. The result of his calculation can also be represented in the form of (5.1), with

$$\frac{\tilde{\mu}_{DSC}}{\mu^{(2)}} = \frac{1}{(c^{(2)})^{g(n)/n}}, \tag{5.18}$$

where $g(n)$ is given in the above reference. From the plot of this function, we obtain that $g(1) = 5/2$, $g(3) \approx 3.3$ and $g(10) \approx 6.6$.

As before our new results reduce to the linear results in the limit as n approaches unity. Figure 5 gives results for the above estimates of the effective modulus for the two usual values of n (3 and 10). Note that the plot is given in terms of $\mu^{(2)}/\tilde{\mu}$ as a function of $c^{(2)}$, so that $\mu^{(2)}/\tilde{\mu}$ tends to zero as $c^{(2)}$ tends to unity. Our SC estimate is slightly lower than the prior result of PCW; this is probably due to the fact that if we had an upper bound, it would also appear lower in our plot. On the other hand, our

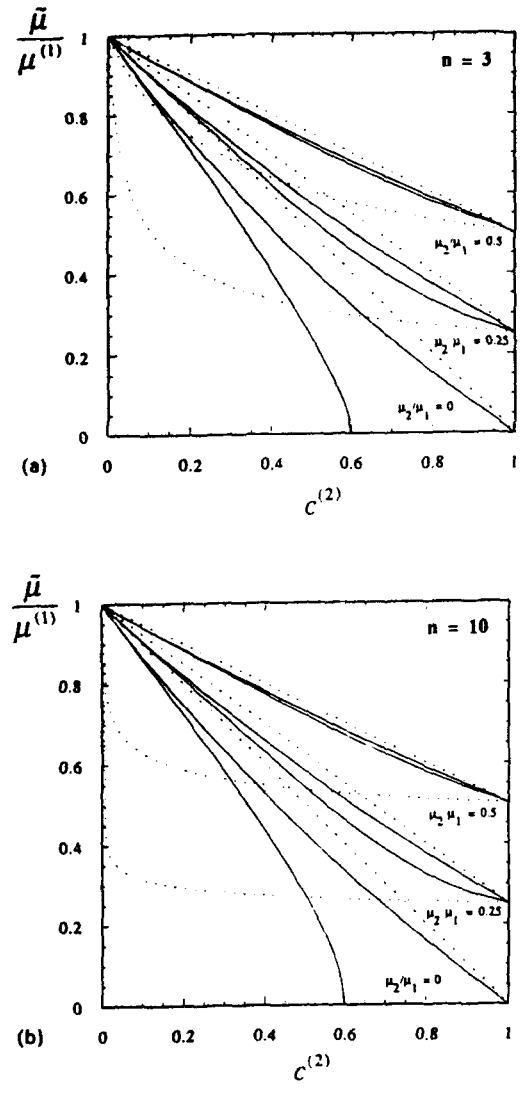


FIG. 4. Bounds and estimates for the effective modulus of the two-phase incompressible composite as functions of the volume fractions of phase #2 for three different ratios of the moduli of the two phases. The continuous lines correspond to the *new* HS upper bound and SC estimate, and the dotted lines correspond to the Voigt/Reuss bounds. Case (a) is for $n = 3$, and (b) for $n = 10$.

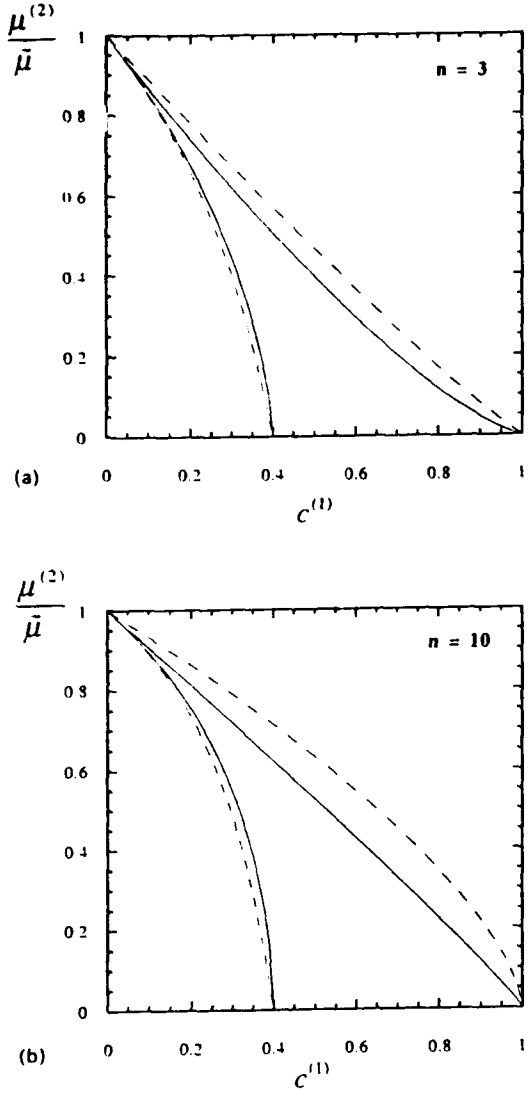


FIG. 5. Estimates of the effective modulus of the rigidly reinforced composite as functions of the volume fraction of the rigid phase. The continuous lines correspond to the *new* SC and DSC estimates, and the dash lines correspond to the SC estimate of PCW, and the DSC estimate of DUVA (1984). Case (a) is for $n = 3$, and (b) for $n = 10$.

DSC estimate lies somewhat above the DSC of DUVA (1984). This is not too surprising since we expect our procedure in general to overestimate the energy of the composite $\tilde{W}(\tilde{\varepsilon})$. For the linear case, it has been shown (MILTON, 1985; AVELLANEDA, 1987) that the DSC estimates can be attained by particular microstructures. Then, we could expect our procedure to give an upper bound for the actual $\tilde{W}(\tilde{\varepsilon})$ of the corresponding nonlinear material. Thus, the present example could be used to assess the efficiency of our method in estimating the exact effective properties of nonlinear composites with deterministic microstructure, such as periodic composites. This idea will be tested elsewhere, but if the indications of the present example are upheld, we would conclude that our prescription performs rather well given its simplicity.

6. CONCLUDING REMARKS

The main contribution of this work is the establishment of a new variational structure that allows the estimation of the effective properties of nonlinear composites in terms of the corresponding properties for linear composites with the same microstructural distribution of phases. Assuming that the exact effective energy density is available for the linear composite, and depending on the growth conditions of the phase potentials, the new estimate will either be an upper bound, or a lower bound for the actual effective energy density of the nonlinear composite. Alternatively, if only an estimate or a bound of the right type (an upper bound if the estimate is an upper bound, or vice versa) is available for the linear composite, the new estimate for the nonlinear composite will also be either only an estimate, or a bound (of the same type). In this respect, the present variational structure has the same limitation as the structure proposed by TW in that only one-sided bounds will result in general.

In the context of specific results, we note that the nonlinear bounds, corresponding to the linear Voigt bounds, proposed by the new prescription turn out to be precisely the nonlinear Voigt bounds obtained directly from the principle of minimum potential energy. On the other hand, the nonlinear bounds corresponding to the Hashin–Shtrikman bounds for the linear isotropic composite turn out to be superior in some cases to the Talbot–Willis bounds for the same material, and identical in other cases (see PONTE CASTAÑEDA, 1990, for an example). Finally, whereas the Talbot–Willis structure leads to some ambiguity in the prescription of self-consistent estimates, the new structure leads to a unique prescription for a self-consistent estimate, as well as to a straightforward generalization of other types of estimates, including differential and dilute estimates. The main advantages of the new structure are the simplicity of its implementation and the generality of its potential applications. Clearly, these are features that could make the proposed structure of great practical, as well as theoretical value. An alternative derivation of this structure is given in PONTE CASTAÑEDA (1990), and some of the potential applications to other types of composites will be addressed elsewhere.

ACKNOWLEDGEMENTS

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APPENDIX: EXACT DUALITY BETWEEN f AND $V^{(1)}$

Here we demonstrate that

$$f(s) = \sup_{\mu^{(1)} > 0} \left\{ \frac{1}{6\mu^{(1)}} s^2 - V^{(1)}(\mu^{(1)}) \right\}. \tag{A1}$$

given that

$$V^{(1)}(\mu^{(1)}) = \sup_{s > 0} \left\{ \frac{1}{6\mu^{(1)}} s^2 - f(s) \right\}. \tag{A2}$$

Proof: Let $x = s^2$, and assume that $F(x) = f(s)$ is a convex function of its argument. Then, if we define $G(y)$ via the Legendre transform

$$G(y) = \sup_{x > 0} \{xy - F(x)\}, \tag{A3}$$

we have by Fenchel duality (VAN TIEL, 1984, Corollary to Section 6.11a) that

$$F(x) = \sup_{y > 0} \{xy - G(y)\}. \tag{A4}$$

Now, if we let $y = 1/6\mu^{(1)}$, (A2) and (A3) imply that

$$G(y) = V^{(1)}(\mu^{(1)}), \tag{A5}$$

and therefore it follows from (A4) that (1) holds.

This result is a special case of a more general result derived in PONTE CASTAÑEDA (1990).

NOTE ADDED IN PROOF

Following the preparation of this paper, Professor J. R. Willis has shown (*J. Mech. Phys. Solids* **39**, 73–86, 1990: this issue) that, with an optimal choice of the comparison material, the Talbot–Willis generalization of the Hashin–Shtrikman variational principle to nonlinear problems can be used to reproduce the new Hashin–Shtrikman bounds obtained in this paper via the new method. However, the new method is more general in the sense that other higher-order bounds and estimates for the linear problem could be used to yield new higher-order bounds and estimates for the corresponding nonlinear problem.

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The Effective Properties of Brittle/Ductile Incompressible Composites

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Abstract

A new variational method for estimating the effective properties of nonlinear composites in terms of the corresponding properties of linear composites with the same microstructural distributions of phases is applied to an isotropic, incompressible composite material containing a brittle (linear) and a ductile (nonlinear) phase. More specifically, in this particular work the prescription is used to obtain bounds of the Hashin-Shtrikman type for the effective properties of the nonlinear composite in terms of the well-known linear bounds. It can be shown that in some cases the method leads to optimal bounds.

Introduction

PONTE CASTAÑEDA (1990a) has proposed a new procedure for estimating the effective properties of composite materials with phases exhibiting nonlinear constitutive behavior. The procedure, which is straightforward to implement, expresses the effective properties of the nonlinear composite in terms of the effective properties of a family of linear composites with the same distribution

Reference [10]

of phases as the nonlinear composite. Appropriate references for the linear theory of composites are given by the review article of WILLIS (1982) and by the monograph of CHRISTENSEN (1979). The new procedure was applied in the above reference to materials containing a nonlinear matrix either weakened by voids or reinforced by rigid particles. Estimates and rigorous bounds were obtained for the effective properties of such materials. The Hashin-Shtrikman bounds (obtained via the new method from the linear Hashin-Shtrikman bounds) were found to be an improvement over the corresponding bounds obtained by PONTE CASTAÑEDA and WILLIS (1988) for the same class of materials using an extension of the Hashin-Shtrikman variational principle to nonlinear problems proposed by TALBOT and WILLIS (1985). Recently, WILLIS (1990) has shown that the Hashin-Shtrikman bounds obtained via the new method can also be obtained by the method of TALBOT and WILLIS (1985) with an optimal choice of the comparison material generally, however, the new procedure can make use of linear higher-order bounds and estimates to yield corresponding bounds and estimates for nonlinear materials. In fact, the new procedure can be shown to yield exact results for a certain class of nonlinear composites. This is discussed in detail by PONTE CASTAÑEDA (1990b).

In this paper we apply the general procedure to a composite containing a brittle (linear) and a ductile (nonlinear) phase. We assume that the phases are perfectly bonded to each other, incompressible and isotropic. Additionally, the size of the typical heterogeneity is assumed to be small compared to the size of the specimen and the scale of variation of the applied loads. It is further assumed that the effect of the interfaces is negligible, so that the effective properties of the composite are essentially derived from the bulk behavior of the constituent phases. Both upper and lower bounds of the Hashin-Shtrikman type are given for the isotropic composite as functions of the properties and volume fractions of the phases. Specific results are given when the behavior of the nonlinear phase is linear plus power-law, including the pure power-law case. Some of the bounds are shown to be *optimal* (i.e., microstructures can be given attaining these bounds).

Effective Properties

Consider a two-phase composite occupying a region of unit volume Ω, with each phase occupying a subregion $\Omega^{(r)}$ ($r = 1, 2$), and let the stress potential, $U(\boldsymbol{\sigma},\mathbf{x})$, be expressed in terms of the homogeneous phase potentials, $U^{(r)}(\boldsymbol{\sigma})$, via

$$U(\boldsymbol{\sigma},\mathbf{x}) = \sum_{r=1}^{2} \chi^{(r)}(\mathbf{x}) U^{(r)}(\boldsymbol{\sigma}), \tag{1}$$

where

$$\chi^{(r)}(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in \Omega^{(r)} \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

is the characteristic function of phase r. The phases are assumed to be incompressible and isotropic, so that the potentials $U^{(r)}(\boldsymbol{\sigma})$ can be assumed to depend on the stress $\boldsymbol{\sigma}$ only through the effective stress

$$\sigma_e = \sqrt{\frac{3}{2}\mathbf{S}\cdot\mathbf{S}},$$

where $\mathbf{S}$ is the deviator of $\boldsymbol{\sigma}$. Thus, we assume that there exist scalar-valued functions $f^{(r)}$ such that

$$U^{(r)}(\boldsymbol{\sigma}) = f^{(r)}(\sigma_e).$$

Then, the stress field $\boldsymbol{\sigma}$, satisfying the equilibrium equations

$$\sigma_{ij,j} = 0, \tag{3}$$

is related to the strain field $\boldsymbol{\varepsilon}$, related to the displacement field $\mathbf{u}$ via

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \tag{4}$$

through the constitutive relation

$$\varepsilon_{ij} = \frac{\partial U}{\partial \sigma_{ij}}(\boldsymbol{\sigma},\mathbf{x}). \tag{5}$$

The commas in equations (3) and (4) denote differentiation, and the summation convention has also been used in equation (3). We assume that the phases are perfectly bonded, so that the displacement is continuous across the interphase boundaries. However, the strains and, therefore, the stresses may be discontinuous across such boundaries, and hence equation (3) must be interpreted in a weak sense, requiring continuity of the traction components of the stress across the interphase boundaries.

We note that if we let $\mathbf{\varepsilon}$ represent the rate-of-deformation tensor and $\mathbf{u}$ the velocity field, the above equations can be used to model high-temperature creep, as well as high-rate viscoplastic deformations. Here we will present our work in the context of time-independent plasticity (deformation theory), but in view of the above comment the results could be given appropriate interpretations in nonlinear creep and viscoplasticity.

To define the effective properties of the heterogeneous material we introduce, following HILL (1963), the uniform constraint boundary condition

$$\sigma_{ij} n_j = \overline{\sigma}_{ij} n_j, \quad \mathbf{x} \in \partial\Omega, \tag{6}$$

where $\partial\Omega$ denotes the boundary of the composite, $\mathbf{n}$ is its unit outward normal, and $\overline{\boldsymbol{\sigma}}$ is a given constant symmetric tensor. Then, the average stress is precisely $\overline{\boldsymbol{\sigma}}$, i.e.

$$\bar{\sigma} = \int_\Omega \sigma(x) dV \tag{7}$$

and we *define* the average strain in a similar manner by

$$\bar{\varepsilon} = \int_\Omega \varepsilon(x) dV . \tag{8}$$

The effective behavior of the composite, or the relation between the average stress and the average strain then follows from the principle of minimum complementary energy, which can be stated in the form

$$\tilde{U}(\bar{\sigma}) = \min_{\sigma \in S(\bar{\sigma})} \bar{U}(\sigma), \tag{9}$$

where

$$\bar{U}(\sigma) = \int_\Omega U(\sigma, x) dV$$

is the complementary energy functional of the problem,

$$S(\bar{\sigma}) = \{\sigma | \sigma_{ij,j} = 0 \text{ in } \Omega, \text{ and } \sigma_{ij} n_j = \bar{\sigma}_{ij} n_j \text{ on } \partial\Omega\}$$

is the set of statically admissible stresses, and where we have assumed convexity of the nonlinear potential $U(\sigma, x)$. Thus, we have that

$$\bar{\varepsilon}_{ij} = \frac{\partial \tilde{U}}{\partial \sigma_{ij}}(\bar{\sigma}). \tag{10}$$

Our task will be to determine bounds and estimates for $\tilde{U}(\bar{\sigma})$, which, under the above assumptions, is known to be convex.

Bounds and Estimates

A new variational principle for determining bounds and estimates for the effective properties of nonlinear composites in terms of the effective properties of linear composites was proposed by PONTE CASTAÑEDA (1990a,b). In this section, we specialize the derivation given in PONTE CASTAÑEDA (1990b) for the case where both phases are incompressible, and phase #2 is linear so that

$$U^{(2)}(\sigma) = \frac{1}{6\mu^{(2)}} \sigma_e^2 .$$

The new variational principle is based on a representation of the potential of the nonlinear material in terms of the potentials of a family of linear *comparison* materials. Thus, for a homogeneous nonlinear material with "stronger than quadratic" growth in its potential, $U(\sigma)$, we have that

$$U(\sigma) \geq \max_{\mu > 0} \{U_e(\sigma) - V(\mu)\}, \tag{11}$$

where

$$V(\mu) = \max_{\sigma} \{U_e(\sigma) - U(\sigma)\} \tag{12}$$

and

$$U_e(\sigma) = \frac{1}{6\mu} \sigma_e^2 \tag{13}$$

is the the potential of the comparison linear material.

To demonstrate this result, let

$$U(\sigma) = \phi(s), \tag{14}$$

where $s = \sigma_e^2$. Then, the Legendre-Fenchel transform of the scalar-valued function ϕ is given by

$$\phi^*(\alpha) = \max_{s > 0} \{\alpha s - \phi(s)\}, \tag{15}$$

where α is assumed to be positive. A well-known result in convex analysis (VAN TIEL 1984, §6.3) is that

$$\phi(s) \geq \max_{\alpha > 0} \{s\alpha - \phi^*(\alpha)\}, \tag{16}$$

with equality if ϕ is a convex function of its argument. With the identifications $s = \sigma_e^2$ and $\alpha = (6\mu)^{-1}$, we can see that (11) and (12) are but simple restatements of (16) and (15), respectively. In particular,

$$V(\mu) = \phi^*\left(\frac{1}{6\mu}\right). \tag{17}$$

To derive the new variational principle, we apply (11) to the nonlinear phase #1, and make use of the result in the complementary energy principle (9). Thus, after some manipulations, we find that

$$\tilde{U}(\bar{\sigma}) \geq \max_{\mu^{(1)}(x)} \left\{ \tilde{U}_e(\bar{\sigma}) - \int_{\Omega^{(1)}} V^{(1)}(\mu^{(1)}) dV \right\}, \tag{18}$$

where

$$\tilde{U}_e(\bar{\sigma}) = \min_{\sigma \in S(\bar{\sigma})} \bar{U}_e(\sigma), \tag{19}$$

$$U_e(\sigma, x) = \sum_{r=1}^{2} \chi^{(r)}(x) U_e^{(r)}(\sigma),$$

and

$$U_e^{(r)}(\sigma) = \frac{1}{6\mu^{(r)}} \sigma_e^2 .$$

Note that the comparison linear material agrees with the actual material in phase # 2 (which is linear). In the above derivation, we note that the comparison moduli $\mu^{(1)}$ are functions of x, since the stress field σ will also in general be a function of x within phase #1. If we assume that $U^{(1)}(\sigma)$ is "strongly convex" (i.e. if ϕ is convex), then we have equality in (11), and hence, usually, equality in (18). However, if the conditions for equality are not met, relation (18) still provides a useful lower bound for $\tilde{U}(\overline{\sigma})$. An detailed derivation of this result, discussing the precise conditions for equality, is given in PONTE CASTAÑEDA (1990b).

The variational principle described by (18) roughly corresponds to solving a completely linear problem for a heterogeneous material with arbitrary moduli variation within the nonlinear phase, and then optimizing with respect to the variations in moduli within the nonlinear phase. Thus, one can think of the nonlinear material as a "linear" material with variable moduli that are determined by prescription (18) in such a way that its properties agree with those of the nonlinear material.

This suggests that if the fields happen to be constant over the nonlinear phase, then the variable moduli $\mu^{(1)}(x)$ can be replaced by constant moduli $\mu^{(1)}$. More generally, however, we have the following lower bound for $\tilde{U}(\overline{\sigma})$

$$\tilde{U}_-(\overline{\sigma}) = \max_{\mu^{(1)}>0}\{\tilde{U}_0(\overline{\sigma}) - c^{(1)}V^{(1)}(\mu^{(1)})\}, \tag{20}$$

where $c^{(1)}$ is the volume fraction of phase #1. The result in this form is a special case of a more general result first derived by PONTE CASTAÑEDA (1990a), when only one of the phases is nonlinear, and the other one is linear.

We note that the prescriptions (18) and (20) lead to convex expressions for the bounds and est[illegible] of the effective potential of the nonlinear composite, provided that the c[illegible]nding bounds and estimates for the linear composite are convex. This is a desirable feature, because the effective potential of the composite is known to be convex.

Application to Hashin-Shtrikman Bounds

HASHIN and SHTRIKMAN (1962) prescribed bounds for the effective moduli of linear-elastic, isotropic composites, depending only on the volume fractions of the phases. When there are only two phases, these bounds have been shown to be optimal (i.e., microstructures can be given that simultaneously attain the bounds for the shear and bulk modulus) by FRANCFORT and MURAT (1987).

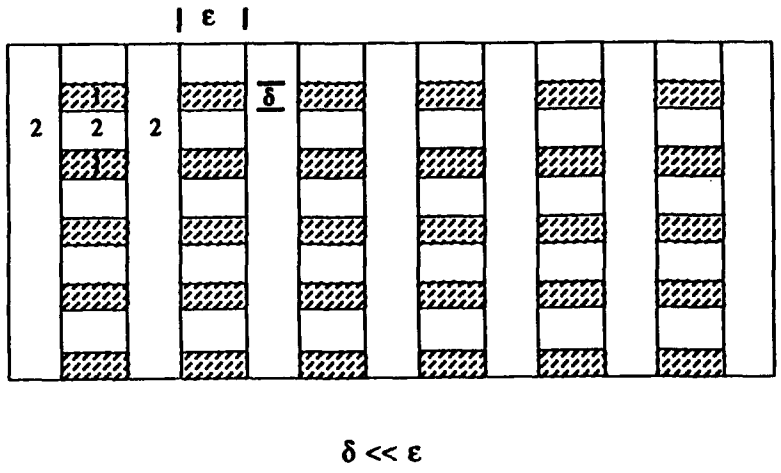

Figure 1. Rank-2 laminate.

Their construction made use of *iterated* laminates for which the effective properties can be computed exactly. Such materials are obtained by layering the two constituent phases to obtain a rank-1 laminate; the resulting material is once again layered (in an arbitrary direction) with one of the original phases in a smaller lengthscale. This procedure can obviously be iterated n times to obtain a rank-n laminate. In general such materials will be anisotropic, but by choosing appropriately the layer orientations at the different layering operations, it is possible to obtain an isotropic composite, and its properties coincide with one of the Hashin-Shtrikman (H-S) bounds depending on which constituent phase is selected to play the role of the matrix material. Figure 1 depicts a rank-2 laminate (not to scale) with phase #2 as the matrix phase.

For the special case of incompressible materials, when there is only one modulus for the composite, the H-S upper bound for the effective shear modulus can be expressed in the form

$$\tilde{\mu}_+ = \begin{cases} \dfrac{\mu^{(1)}}{\alpha(\mu^{(1)},\mu^{(2)})} & \text{if } \mu^{(1)} \geq \mu^{(2)} \\ \dfrac{\mu^{(2)}}{\beta(\mu^{(1)},\mu^{(2)})} & \text{if } \mu^{(1)} \leq \mu^{(2)} \end{cases}, \tag{21}$$

where

$$\alpha(\mu^{(1)},\mu^{(2)}) = \frac{2c^{(1)}\mu^{(2)} + (3+2c^{(2)})\mu^{(1)}}{(2+3c^{(2)})\mu^{(2)} + 3c^{(1)}\mu^{(1)}} \tag{22}$$

and

$$\beta(\mu^{(1)},\mu^{(2)}) = \frac{2c^{(2)}\mu^{(1)} + (3+2c^{(1)})\mu^{(2)}}{(2+3c^{(1)})\mu^{(1)} + 3c^{(2)}\mu^{(2)}} \quad . \tag{23}$$

The corresponding H-S lower bound is obtained by interchanging the expressions in (21) for the upper bound (and keeping the conditions on the shear moduli $\mu^{(1)}$ and $\mu^{(2)}$ fixed).

The above H-S upper bound for the effective shear modulus yields a lower bound for the potential of the linear material $\tilde{U}_*$. This information can be used in combination with prescription (20) to yield a H-S lower bound for the potential of the nonlinear material $\tilde{U}$. On the other hand, upper bounds for $\tilde{U}_*$ do not necessarily generate upper bounds for $\tilde{U}$.

The result for the lower bound on $\tilde{U}$ depends on which of the two branches of (21) is used in conjunction with (20). If $\mu^{(1)} > \mu^{(2)}$, then the average effective stress $\overline{\sigma}_*$ must be such that the condition

$$3\mu^{(2)} f'(\overline{\sigma}_*) < \overline{\sigma}_*, \tag{24}$$

is satisfied (usually when the average shear stress is small enough). Here, for simplicity, we have made the identification $f^{(1)} = f$. The corresponding form of the bound is then

$$\tilde{U}_-(\overline{\sigma}) = \tilde{f}_1(\overline{\sigma}_*), \tag{25}$$

where

$$\tilde{f}_1(\overline{\sigma}_*) = c^{(1)} f(s) + \mu^{(2)}\left[\frac{(2+3c^{(2)})}{2} - \left(\frac{\overline{\sigma}_*}{s}\right)^2\right](f'(s))^2 \tag{26}$$

and s solves the equation

$$c^{(1)} + \mu^{(2)}(2+3c^{(2)})\frac{f'(s)}{s} = \frac{5}{3}\left[\frac{(2+3c^{(2)})(s/\overline{\sigma}_*)^2 - 2}{3c^{(2)}}\right]^{-1/2}. \tag{27}$$

On the other hand, if $\mu^{(1)} < \mu^{(2)}$, then the average effective stress $\overline{\sigma}_*$ must be such that the condition

$$3\mu^{(2)} f'(\overline{\sigma}_*) > \overline{\sigma}_*, \tag{28}$$

is satisfied (i.e., when the average shear stress is large enough), and

$$\tilde{U}_-(\overline{\sigma}) = \tilde{f}_2(\overline{\sigma}_*), \tag{29}$$

where

$$\tilde{f}_2(\overline{\sigma}_*) = c^{(1)} f(s) + \frac{(3+2c^{(1)})\overline{\sigma}_*^2 + c^{(1)}(2+3c^{(1)})s^2 - 10c^{(1)}s\overline{\sigma}_*}{18c^{(2)}\mu^{(2)}}, \tag{30}$$

and s solves the equation

$$9c^{(2)}\mu^{(2)} f'(s) = 5\overline{\sigma}_* - (2+3c^{(1)})s. \tag{31}$$

The corresponding stress/strain relations have the form

$$\mathbf{E} = \frac{3}{2}\tilde{f}'(\overline{\sigma}_*)\mathbf{S}, \tag{32}$$

where

$$\tilde{f}_2'(\overline{\sigma}_*) = \frac{2}{3}\frac{(3+2c^{(1)})\overline{\sigma}_* - 5c^{(1)}s}{6c^{(2)}\mu^{(2)}}, \tag{33}$$

but $\tilde{f}_1'(\overline{\sigma}_*)$ does not have a simple expression.

In general, we do not expect the above lower bounds for $\tilde{U}$ to be optimal. In fact, expression (25) does not yield an optimal bound if condition (24) is satisfied. However, it is shown in PONTE CASTAÑEDA (1990b) that if condition (28) is satisfied, then the bound (29) is optimal. This is because the same microstructure attaining the linear bounds can be also shown to attain the nonlinear bound; the reason being that the fields are constant in the (nonlinear) inclusion phase, and hence expressions (20) and (18) are identical. Similar observations have been made by KOHN (1990) in a similar context (starting from the Talbot-Willis nonlinear variational principle) and, independently, by PONTE CASTAÑEDA (1990c) in the context of conductivity.

Conversely, in general, we do not expect that interchanging conditions (24) and (28) would turn expression (25) and (29) into upper bounds for the nonlinear potential $\tilde{U}$. This is contrary to the corresponding operation for the linear composite. All that can be said, however, is that expression (29) is an estimate for the upper bound for $\tilde{U}$ if condition (24) is satisfied and that expression (25) is an estimate for the upper bound for $\tilde{U}$ if condition (28) is satisfied. Both of these estimates are expected to get progressively better with weaker nonlinearities.

Application to Power-Law Behavior

In this section, we specialize further the calculations of the previous section by taking the constitutive behavior of the nonlinear phase to be governed by a linear plus power relation

$$f(\sigma_e)=\frac{1}{3}\left[\frac{1}{2\mu}+\left(\frac{1}{n+1}\right)\frac{1}{\eta}\sigma_e^{n-1}\right]\sigma_e^2. \tag{34}$$

Note that the case $\mu\to\infty$ corresponds to pure power-law behavior, and the limits $n\to 1$ (in addition to $\mu\to\infty$) or $\eta\to\infty$ correspond to linear behavior.

The conditions (24) and (28) determining the appropriate branch of the bound specialize to

$$\frac{\mu^{(2)}}{\mu}+\frac{\mu^{(2)}}{\eta}\bar{\sigma}_e^{n-1}<1 \tag{35}$$

and the opposite inequality, respectively. The first condition guaranteeing that (25) is a lower bound (and correspondingly that (29) is an estimate for the upper bound) corresponds to small enough average stress on the composite. Alternatively, the second condition (with > instead of <) corresponds to sufficiently large average stress. Note that, if $\mu^{(2)}/\mu>1$, condition (35) can never be satisfied and, conversely, the alternative condition is always satisfied. This condition ensures that the difference between the potential of phase #1 and that of phase #2 is convex. Here, we will consider two cases: one case, meeting this condition, with $\mu^{(2)}/\mu=2$, and the other with $\mu^{(2)}/\mu=0$, corresponding to the pure power-law case.

The results for the bounds (25) and (29) specialized to the case when (34) holds can be expressed in the form:

$$\frac{\tilde{U}(\bar{\sigma})}{U^{(2)}(\bar{\sigma})}=F\left\{\frac{\mu^{(2)}}{\eta}\bar{\sigma}_e^{n-1};\frac{\mu^{(2)}}{\mu},c^{(2)},n\right\}, \tag{36}$$

where the precise form of F depends on whether (25) or (29) applies, and $(\mu^{(2)}/\eta)\bar{\sigma}_e^{n-1}$ plays the role of the independent variable, with $\mu^{(2)}/\mu$, $c^{(2)}$ and n, serving as parameters.

Results for the upper and lower bounds for $\tilde{U}$ are given in Figures 2 and 3 for the case where $\mu^{(2)}/\mu=0$, and in Figure 4 for the case where $\mu^{(2)}/\mu=2$. In the first case, condition (35) determining whether (29) is an estimate for the upper bound or an optimal lower bound, and whether (25) is an estimate for the upper bound, or a non-optimal lower bound, simply reduces to the condition of whether the independent variable $(\mu^{(2)}/\eta)\bar{\sigma}_e^{n-1}$ is less or greater than unity. For that reason, we give results emphasizing the small stress and large stress domains, separately, in Figures 2 and 3, respectively.

In each plot we have three sets of curves corresponding to three values of $c^{(2)}$ (0.1, 0.5 and 0.9). Additionally, we show the limiting cases corresponding to $c^{(2)}=0$ and $c^{(2)}=1$. These limiting curves appear as straight lines, one with with variable slope depending on the value of n and $\mu^{(2)}/\mu$, and the other with

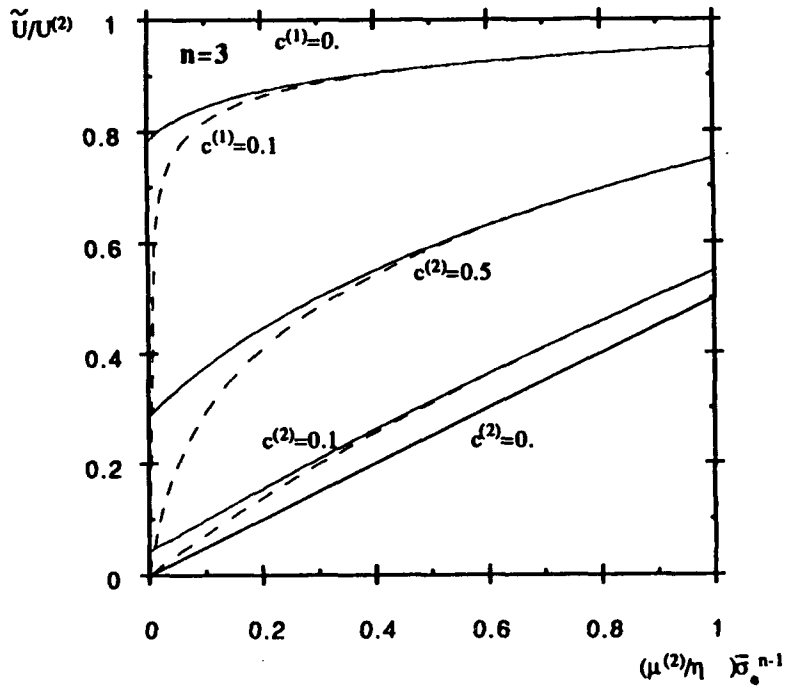

Figure 2(a). Plots of the bounds for the effective energy of the composite as functions of the average stress (appropriately normalized) for $\mu^{(2)}/\mu = 0$ and n = 3 (small stress).

zero slope (value equal to unity), respectively. The intermediate sets of curves correspond to the upper and lower bounds.

In Figure 2, depicting results for two values of the nonlinearity parameter (n = 3 and 10), the continuous line corresponds to the estimate for the upper bound (for $\tilde{U}$), and the dashed line corresponds to the rigorous lower bound. In Figure 3, showing also results for the same two values of the nonlinearity parameter, the continuous line corresponds to the optimal lower bound, and the dashed line is an estimate for the upper bound. For this value of $\mu^{(2)}/\mu$, the upper and lower bound coalesce when the value of the independent variable

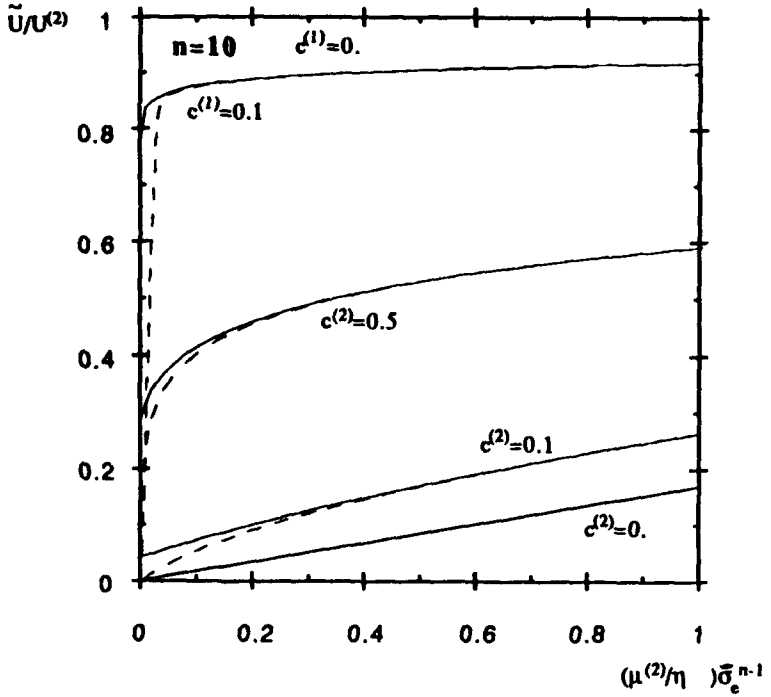

Figure 2(b). Plots of the bounds for the effective energy of the composite as functions of the average stress (appropriately normalized) for $\mu^{(1)}/\mu = 0$ and n = 10 (small stress).

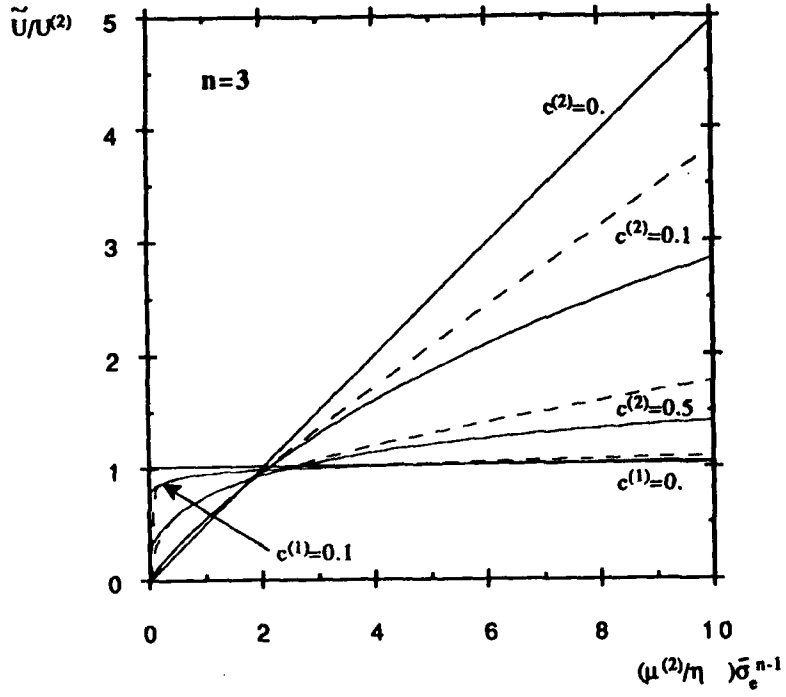

Figure 3(a). Plots of the bounds for the effective energy of the composite as functions of the average stress (appropriately normalized) for $\mu^{(1)}/\mu = 0$ and n = 3 (large stress)

$(\mu^{(2)}/\eta)\bar{\sigma}_e^{n-1}$ approaches unity. In the linear case, this behavior corresponds to the limit of the moduli of the phases approaching each other. More generally, assuming that $\mu^{(2)}/\mu$ is less than unity, there is a value of the independent variable (i.e., an average stress level) at which the bounds are equal, and hence the effective energy of the composite is known exactly. This phenomenon is related to the lack of convexity of the difference between the potentials of the nonlinear and linear phases.

In Figure 4, depicting results for the same two values of the nonlinearity parameter, the continuous line corresponds to the optimal lower bound (for $\tilde{U}$), and the dashed line corresponds to the estimate for the upper bound. In this case, with a convex difference between the nonlinear and linear potentials, there is no value of the independent variable for which the upper and lower bound are equal.

Both in Figures 3 and 4, we observe that the lower bound approaches a straight line with zero slope and the upper bound approaches a straight line with slope depending on the value of n (smaller for larger n). This is consistent with the following asymptotic behaviors for the lower and upper bounds

$$\frac{\tilde{U}(\bar{\sigma})}{U^{(2)}(\bar{\sigma})} \approx \frac{\left(1+\tfrac{1}{3}c^{(1)}\right)}{c^{(2)}}, \tag{37}$$

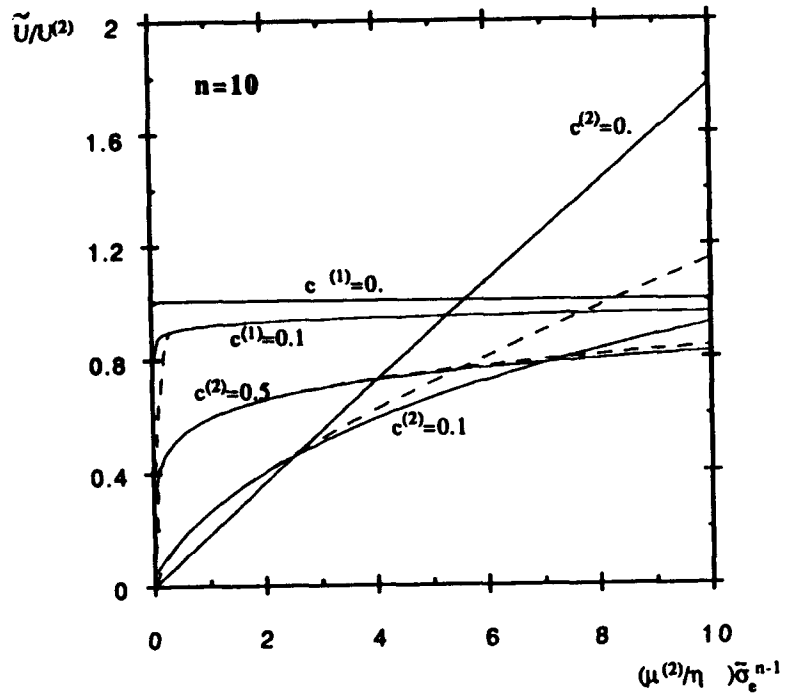

Figure 3(b). Plots of the bounds for the effective energy of the composite as functions of the average stress (appropriately normalized) for $\mu^{(2)}/\mu = 0$ and n = 10 (large stress)

and

$$\frac{\tilde{U}(\bar{\sigma})}{U^{(2)}(\bar{\sigma})} \approx \frac{2}{n+1}\frac{1}{\omega^n}\frac{\mu^{(2)}}{\eta}\bar{\sigma}_e^{n-1}, \tag{38}$$

with

$$\omega = \frac{\left(1+\tfrac{1}{2}c^{(2)}\right)^{\frac{n+1}{n}}}{\left(c^{(1)}\right)^{\frac{1}{n}}} \approx 1+\left(\frac{3n+7}{4n}\right)c^{(2)} \quad \text{as } c^{(2)} \to 0 \tag{39}$$

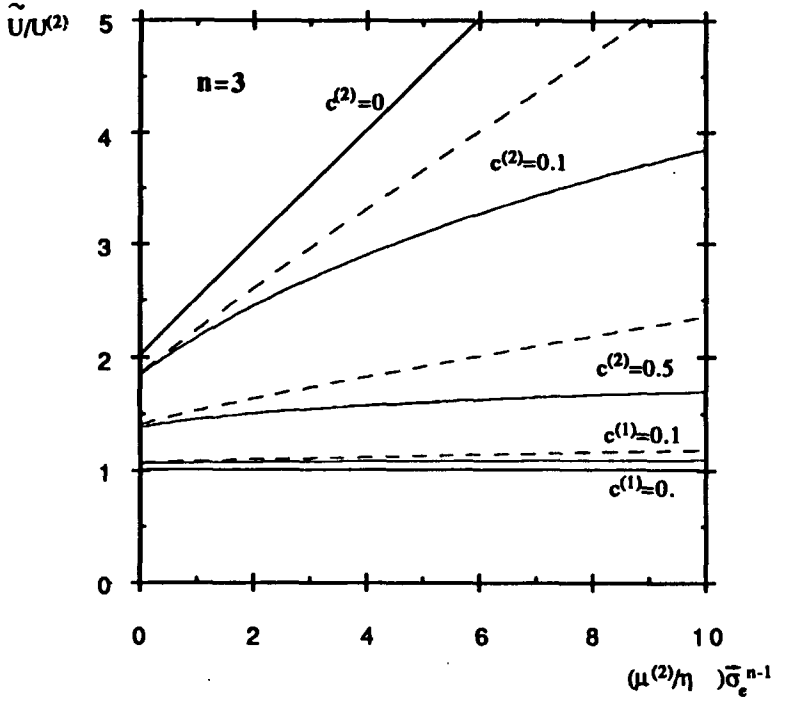

Figure 4(a). Plots of the bounds for the effective energy of the composite as functions of the average stress (appropriately normalized) for $\mu^{(2)}/\mu = 2$ and n = 3.

respectively. These two behaviors correspond physically to the cases of a linear matrix with voids and a power-law matrix with rigid inclusions (studied by PONTE CASTAÑEDA, 1990a), respectively. The reason for these behaviors is that the lower bound (for $\tilde{U}$) corresponds to putting the stiffer material in the matrix phase and the less stiff material in the inclusion phase (and viceversa for the upper bound). Clearly, for large enough stresses, the linear phase is stiffer than the nonlinear phase.

We note that accurate numerical calculations of the potential of a power-law matrix with spherical rigid inclusion have yielded results of the form (38) with

$$\omega = 1+\left(\frac{g(n)}{n}\right)c^{(2)} \quad \text{as } c^{(2)} \to 0, \tag{40}$$

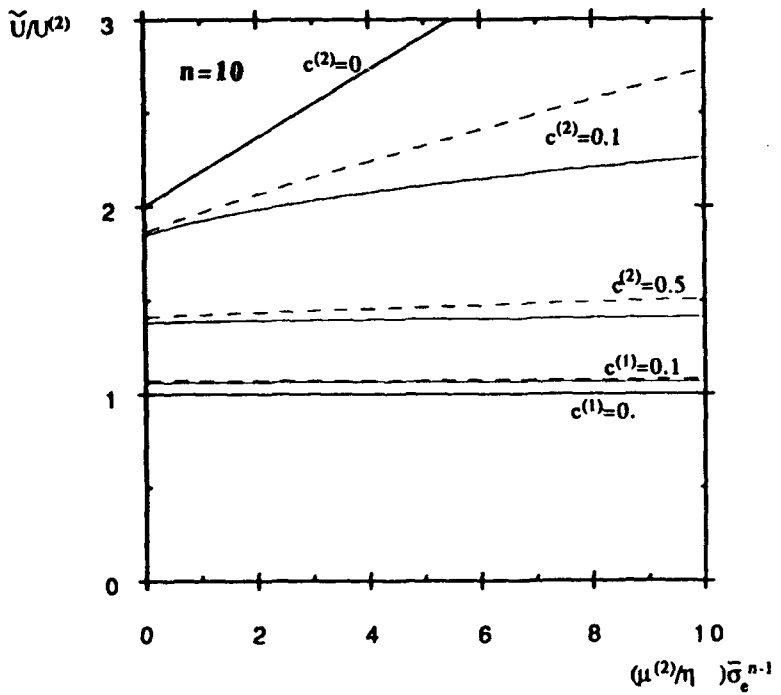

Figure 4(b). Plots of the bounds for the effective energy of the composite as functions of the average stress (appropriately normalized) for $\mu^{(1)}/\mu = 2$ and n = 10.

where $g(n)$ is such that $g(1) = 5/2$, $g(3) \approx 3.21$ and $g(10) \approx 6.09$ (LEE and MEAR, 1990), and $g(n) \to 0.38n$ as $n \to \infty$ (HUTCHINSON, 1990). These results do not compare very favorably with the corresponding results from (39): 5/2, 4.00, 9.25 and $0.75n$, but it should be recalled that these results correspond to the case for which we do not have a rigorous bound (it is simply an estimate of the bound). None the less, the results of (38) with (39) may provide reasonable estimates for larger values of the volume fraction of the linear phase.

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NEW VARIATIONAL PRINCIPLES IN PLASTICITY AND THEIR APPLICATION TO COMPOSITES MATERIALS

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Abstract

A new variational method for bounding the effective properties of nonlinear composites with isotropic phases, proposed by PONTE CASTAÑEDA (*J. Mech. Phys. Solids* **39**, 45, 1991), is given here full variational principle status. Two dual versions of the new variational principle are presented and their equivalence to each other, and to the classical variational principles, is demonstrated. The variational principles are used to determine bounds and estimates for the effective energy functions of nonlinear composites with prescribed volume fractions in the context of plasticity. The classical bounds of Voigt and Reuss for completely anisotropic composites are recovered from the new variational principles and are given alternative, simpler forms. Also, use of a new identity allows the determination of novel, simpler forms for nonlinear Hashin-Shtrikman bounds and estimates for isotropic, and transversely isotropic, fiber-reinforced composites. Additionally, third-order bounds of the Beran type are determined for the first time for nonlinear composites. The question of optimality of these bounds is discussed briefly.

1. Introduction

A new variational procedure for bounding the effective properties of nonlinear composites with isotropic phases has been proposed recently by PONTE CASTAÑEDA (1991a). The procedure allows the estimation of the effective properties of nonlinear composites in terms of the effective properties of an appropriately optimized, linear *comparison composite* with the same underlying microstructures as the nonlinear composite. The general nonlinear estimate can be expressed in the form of a lower bound for the effective complementary-energy function of the composite, in such a way that if a lower bound is available for the linear comparison composite, a lower bound can be deduced for the nonlinear composite. On the other hand, an upper bound for the effective complementary-energy function of the comparison composite does not yield in general an upper bound for the nonlinear composite. In this sense, the new procedure is similar to a nonlinear generalization of the Hashin-Shtrikman procedure, developed by TALBOT and WILLIS (1985) following WILLIS (1983), which makes use of a linear *homogeneous* comparison material to obtain bounds and estimates of the Hashin-Shtrikman type for nonlinear composites. The new variational procedure, however, allows the use of *any* linear bound, or estimate for the comparison composite to yield a corresponding nonlinear bound, or estimate for the nonlinear composite.

In this paper, we strengthen the variational procedure proposed by PONTE CASTAÑEDA (1991a) to a full variational principle, which under appropriate hypothesis can be shown to be completely equivalent to the classical minimum principles of potential and complementary energy. This goal is accomplished essentially by allowing the comparison composite to have continuously varying microstructure, instead of being constrained to have the same (piecewise constant) microstructure as the nonlinear composite. Naturally, the v ational procedure of PONTE CASTAÑEDA (1991a) results as an *approximation*, but more generally, the new variational principles allows the computation of the *exact* properties of certain classes of special microstructures. For example, DE BOTTON and PONTE CASTAÑEDA (1991) have made use of the new variational principles to determine simple expressions for the effective properties of nonlinear

laminates. Here, we will make use of the *exact* version of the new variational principles to demonstrate the optimality of some of the nonlinear bounds developed by means of the *approximate* version of the variational principle.

The paper is organized as follows. First, the definition of the effective properties of nonlinear composites is reviewed. Then, two dual versions of the new variational principle are presented, with rigorous proofs of their respective equivalence to each other, and to the classical variational principles. In the following section, we apply the *approximate* version of the new variational principle, together with a new identity introduced in DE BOTTON and PONTE CASTAÑEDA (1991), to develop remarkably *simple* forms for the Voigt and Hashin-Shtrikman lower bounds for the effective complementary-energy function of nonlinear composites with phases in prescribed volume fractions. Results are given for multiple-phase, compressible composites, but a particularly simple form for the effective energy of the nonlinear composite is obtained for the case of two-phase incompressible composites. These results are expressed in terms of a one-dimensional optimizations problem [see equation (4.34)], and generalize the findings of PONTE CASTAÑEDA (1991b) for two-phase, incompressible composites with a linear and a nonlinear phase. For the Hashin-Shtrikman bounds, results are given not only for isotropic nonlinear composites, but also for transversely isotropic, fiber-reinforced materials [see equation (4.46)]. Additionally, a *third-order* bound of the Beran type is given for isotropic composites [see equation (4.41)]. To the knowledge of the author, this is the first higher-order bound ever produced in the context of nonlinear composites, and illustrates the versatility of the new variational procedure. Additionally, in this section, we give a derivation of the Reuss upper bound for the effective complementary-energy function of the nonlinear composite by means of the *exact* version of the new variational principle. This serves to demonstrate that the exact version of the variational principle is able to deliver upper bounds (which cannot be obtained from the approximate version of the variational prin). Although we were unable to determine rigorous upper bounds of the Hashin-Shtrikman and n varieties, we produce some arguments justifying the application of the approximate version of the variational procedure in connection with upper bounds for the linear comparison

composite to obtain "upper estimates" for the effective properties of classes of nonlinear composites. Finally, the question of attainability of the bounds is discussed briefly.

In the development of the bounds, we will be guided by the physical requirements of the deformation theory of plasticity, although completely analogous results will also hold in the context of nonlinear creep. Additionally, we could follow the arguments presented by DUVA and HUTCHINSON (1984) to make use of the effective complementary-energy function of a given deformation-theory composite to obtain an *approximate* form for the yield function of a flow-theory composite exhibiting the same microstructure (and coinciding plastic response under proportional loading) as the deformation-theory solid.

2. Effective Properties

We consider a heterogeneous solid occupying a domain of unit volume $\Omega \in \mathrm{R}^3$, with boundary $\partial\Omega$. The nonlinear constitutive behavior of the solid is characterized by an energy density function, $w(\mathbf{x},\boldsymbol{\varepsilon})$, depending on the position vector $\mathbf{x}$ and the strain field $\boldsymbol{\varepsilon}(\mathbf{x})$, in such a way that the stress field $\boldsymbol{\sigma}(\mathbf{x})$ is given by

$$\boldsymbol{\sigma}(\mathbf{x}) = \partial_{\boldsymbol{\varepsilon}} w(\mathbf{x},\boldsymbol{\varepsilon}). \tag{2.1}$$

Here $\partial_{\boldsymbol{\varepsilon}}$ denotes the usual differentiation with respect to $\boldsymbol{\varepsilon}$, if w is differentiable. More generally, $\partial_{\boldsymbol{\varepsilon}}$ can be given the interpretation of the subdifferential of convex analysis (EKELAND and TEMAM, 1974; § I.5).

Following HILL (1963), we define the *effective* constitutive behavior of the heterogeneous solid in terms of the analogous relation

$$\overline{\boldsymbol{\sigma}} = \partial_{\overline{\boldsymbol{\varepsilon}}} \tilde{W}(\overline{\boldsymbol{\varepsilon}}), \tag{2.2}$$

where $\overline{\boldsymbol{\sigma}}$ denotes the mean value of the stress field over Ω, and $\tilde{W}$ refers to the normalized energy (recall that Ω has unit volume) of the solid when subjected to the uniform boundary condition

$$\mathbf{u} = \overline{\boldsymbol{\varepsilon}}\mathbf{x}, \quad \mathbf{x} \in \partial\Omega, \tag{2.3}$$

where $\mathbf{u}$ is the displacement field, and $\bar{\boldsymbol{\varepsilon}}$ is a constant, symmetric tensor. We note that under this boundary condition, the mean value of the strain over Ω is precisely $\bar{\boldsymbol{\varepsilon}}$.

The *effective* energy of the solid, $\tilde{W}$, can be obtained directly in terms of the principle of minimum potential energy via the relation

$$\tilde{W}(\bar{\boldsymbol{\varepsilon}}) = \min_{\boldsymbol{\varepsilon} \in K(\bar{\boldsymbol{\varepsilon}})} W(\boldsymbol{\varepsilon}), \tag{2.4}$$

where

$$W(\boldsymbol{\varepsilon}) = \int_{\Omega} w(\mathbf{x}, \boldsymbol{\varepsilon}(\mathbf{x}))\, dx \tag{2.5}$$

is the pertinent energy functional, and

$$K(\bar{\boldsymbol{\varepsilon}}) = \left\{ \boldsymbol{\varepsilon} \middle| \boldsymbol{\varepsilon} = \frac{1}{2}\left[\nabla \mathbf{u} + (\nabla \mathbf{u})^T\right] \text{ in } \Omega, \text{ and } \mathbf{u} = \bar{\boldsymbol{\varepsilon}}\mathbf{x} \text{ on } \partial\Omega \right\} \tag{2.6}$$

is the set of *kinematically* admissible strain fields.

In plasticity, it is usually more convenient to express the constitutive relation of the solid in terms of a complementary-energy density function, $u = w^*$, where w^* is the Legendre transform, or polar function of convex analysis (EKELAND and TEMAM, 1974; § I.4.1), defined by

$$w^*(\mathbf{x}, \boldsymbol{\sigma}) = \sup_{\boldsymbol{\varepsilon}} \{\boldsymbol{\varepsilon} \cdot \boldsymbol{\sigma} - w(\mathbf{x}, \boldsymbol{\varepsilon})\}. \tag{2.7}$$

Thus, the constitutive relation of the heterogeneous solid in its local form can alternatively be written

$$\boldsymbol{\varepsilon}(\mathbf{x}) = \partial_{\boldsymbol{\sigma}} u(\mathbf{x}, \boldsymbol{\sigma}). \tag{2.8}$$

Correspondingly, the effective constitutive relation of the solid (in its global form) is given by

$$\bar{\boldsymbol{\varepsilon}} = \partial_{\bar{\boldsymbol{\sigma}}} \tilde{U}(\bar{\boldsymbol{\sigma}}), \tag{2.9}$$

where $\tilde{U}$ corresponds to the normalized complementary energy that would exist in the heterogeneous solid when subjected to a uniform constraint on $\partial\Omega$, such that

$$\boldsymbol{\sigma}\mathbf{n} = \bar{\boldsymbol{\sigma}}\mathbf{n}, \tag{2.10}$$

where $\overline{\boldsymbol{\sigma}}$ is a constant, symmetric tensor. Thus, the effective complementary energy of the composite, $\tilde{U}$, can be described in terms of the principle of minimum complementary energy via the relation

$$\tilde{U}(\overline{\boldsymbol{\sigma}}) = \min_{\boldsymbol{\sigma} \in S(\overline{\boldsymbol{\sigma}})} U(\boldsymbol{\sigma}), \tag{2.11}$$

where

$$U(\boldsymbol{\sigma}) = \int_{\Omega} u(\mathbf{x}, \boldsymbol{\sigma}(\mathbf{x}))\, dx \tag{2.12}$$

is the complementary energy functional, and

$$S(\overline{\boldsymbol{\sigma}}) = \left\{ \boldsymbol{\sigma} \,\middle|\, \nabla \cdot \boldsymbol{\sigma} = 0 \text{ in } \Omega, \text{ and } \boldsymbol{\sigma}\mathbf{n} = \overline{\boldsymbol{\sigma}}\mathbf{n} \text{ on } \partial\Omega \right\} \tag{2.13}$$

is the set of *statically* admissible stress fields.

We will assume that w, and therefore u, are convex functions of $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$, respectively, and that they satisfy certain growth conditions to be specified later. Further, by convex duality (EKELAND and TEMAM, 1974; §I.4.2), we have that

$$w(\mathbf{x}, \boldsymbol{\varepsilon}) = \sup_{\boldsymbol{\sigma}} \left\{ \boldsymbol{\varepsilon} \cdot \boldsymbol{\sigma} - w^*(\mathbf{x}, \boldsymbol{\sigma}) \right\}. \tag{2.14}$$

Then, the effective energy functions $\tilde{W}$ and $\tilde{U}$ are convex (see PONTE CASTAÑEDA and WILLIS, 1988), but on the other hand, we have that, in general (WILLIS, 1989),

$$\tilde{W}(\overline{\boldsymbol{\varepsilon}}) \geq \sup_{\overline{\boldsymbol{\sigma}}} \left\{ \overline{\boldsymbol{\varepsilon}} \cdot \overline{\boldsymbol{\sigma}} - \tilde{U}(\overline{\boldsymbol{\sigma}}) \right\}. \tag{2.15}$$

The reason for the inequality is that the two definitions of effective properties (in terms of $\tilde{W}$ and $\tilde{U}$) correspond to different boundary conditions (uniform traction *versus* uniform displacement). However, for a *composite* solid, obtained in the limit as the size of the typical heterogeneity in the solid becomes small compared to the size of the specimen, the two definitions agree, and equality holds in (2.15).

3. The New Variational Principles

In the development of the new variational principles, we will consider two dual formulations of essentially the same result, depending on whether we start from the minimum potential energy formulation (2.4), or from the minimum complementary energy formulation (2.11). In both cases, we will assume that the solid is *locally isotropic*, so that we can write

$$w(\mathbf{x},\boldsymbol{\varepsilon}) = \phi(\mathbf{x};\varepsilon_e,\varepsilon_m), \tag{3.1}$$

where $\phi:\Omega\times R^+\times R^+\to R^+$ is a nonnegative function, that is convex and continuous in its last two arguments, and satisfies the condition that $\phi(\mathbf{x};0,0)=0$ for all $\mathbf{x}$. Here, R^+ is the set of the extended nonnegative reals (R is the set of extended reals), and

$$\varepsilon_m = \frac{1}{3}|\mathrm{tr}\,\boldsymbol{\varepsilon}| \quad\text{and}\quad \varepsilon_e = \sqrt{\frac{3}{2}\mathbf{e}\cdot\mathbf{e}}, \tag{3.2}$$

are the *mean* and *effective* (in their plasticity denotations) strains, respectively, where $\mathbf{e}=\boldsymbol{\varepsilon}-\varepsilon_m\mathbf{I}$ is the deviator strain tensor. We note that the above conditions on ϕ are consistent with the assumed convexity of w, and that the form (3.1) is *not* the most general form for the energy function of a nonlinear isotropic solid (we could also have dependence on the determinant of $\boldsymbol{\varepsilon}$), but this form is still general enough to cover the usual plasticity models of interest here. Also, we emphasize that with the above definitions *both* ε_m and ε_e are nonnegative.

We note here for later reference that under the above hypotheses on w, its polar function can be expressed (see EKELAND and TEMAM, 1974; § I.4.3 for an analogous result, and also Appendix I for a definition of $\phi^*:\Omega\times R^+\times R^+\to R^+$) in terms of

$$w^*(\mathbf{x},\boldsymbol{\sigma}) = \phi^*\left(\mathbf{x};\frac{2}{3}\sigma_e,3\sigma_m\right), \tag{3.3}$$

where

$$\sigma_m = \frac{1}{3}|\mathrm{tr}\,\boldsymbol{\sigma}| \quad\text{and}\quad \sigma_e = \sqrt{\frac{3}{2}\mathbf{s}\cdot\mathbf{s}}, \tag{3.4}$$

are respectively the mean and effective stresses, and $\mathbf{s}=\boldsymbol{\sigma}-\sigma_m\mathbf{I}$ is the deviator stress tensor. Alternatively, we may represent $u=w^*$ by

(3.5) $$u(\mathbf{x},\sigma) = \psi(\mathbf{x};\sigma_e,\sigma_m),$$

where ψ is a nonnegative function, convex in its last two argument, and such that $\psi(\mathbf{x};0,0) = 0$ for all $\mathbf{x}$. These properties follow from simple properties of the polar function (see EKELAND and TEMAM, 1974; § I.4). Finally, we have that $w = u^*$ can also be represented by

(3.6) $$u^*(\mathbf{x},\varepsilon) = \psi^*\left(\mathbf{x};\frac{2}{3}\varepsilon_e,3\varepsilon_m\right).$$

3.1. *Minimum potential energy formulation*

The new variational principle relies on a quadratic change of variables $u = h(\varepsilon)$, with $h:\mathbf{R}^+ \to \mathbf{R}^+$ given by $h(\varepsilon) = \varepsilon^2$. We can then define a function $f:\Omega \times \mathbf{R}^+ \times \mathbf{R}^+ \to \mathbf{R}^+$ such that

(3.7) $$f(\mathbf{x};u_e,u_m) = \phi\left(\mathbf{x};h^{-1}(u_e),h^{-1}(u_m)\right).$$

By the properties of ϕ, f is a nonnegative function with same dependence on $\mathbf{x}$ as ϕ, and such that $f(\mathbf{x};0,0) = 0$. We note further that f is continuous, but not necessarily convex (in its last two arguments). In fact, because we are interested in the application of these results to plasticity, we will usually assume that f is concave, or at least that it has concave growth. Thus, for example, for an incompressible power-law material, $w \sim \varepsilon_e^{\frac{n+1}{n}}$ $(n \geq 1)$, and therefore $f \sim u_e^{\frac{n+1}{2n}}$, which is a concave function, even though w, itself, is convex.

We can then define (see Appendix I, and VAN TIEL, 1984; §7.14) the *concave* polar function $f_*:\Omega \times \mathbf{R}^+ \times \mathbf{R}^+ \to \mathbf{R}^+$ by

(3.8) $$f_*(\mathbf{x};p_e,p_m) = \inf_{u_e,u_m \geq 0}\left\{u_e p_e + u_m p_m - f(\mathbf{x};u_e,u_m)\right\},$$

and, therefore, we have that

(3.9) $$f(\mathbf{x};u_e,u_m) \leq \inf_{p_e,p_m \geq 0}\left\{u_e p_e + u_m p_m - f_*(\mathbf{x};p_e,p_m)\right\},$$

with *equality* if f is concave.

We proceed by defining the functional F induced by f as follows

(3.10) $$F(u_e,u_m) = \int_\Omega f\left(\mathbf{x};u_e(\mathbf{x}),u_m(\mathbf{x})\right)dx,$$

and we note that (see EKELAND and TEMAM, 1974; § IX.2.1, 2) that the (concave) polar of the functional F, given by

$$F_*(p_e,p_m)=\inf_{u_e(x),u_m(x)\geq 0}\left\{\int_\Omega\left[u_e(x)p_e(x)+u_m(x)p_m(x)\right]dx-F(u_e,u_m)\right\}, \tag{3.11}$$

can alternatively be represented in terms of f_* via

$$F_*(p_e,p_m)=\int_\Omega f_*\left(x;p_e(x),p_m(x)\right)dx, \tag{3.12}$$

with an analogous representation for the bipolar function F_{**} in terms of f_{**}.

Thus, rewriting the minimum energy principle (2.4) in the form

$$\tilde{W}(\bar{\varepsilon})=\min_{u_e,u_m\in K'}F(u_e,u_m), \tag{3.13}$$

where K' is the set of admissible fields $u_e(x)$, $u_m(x)$ induced by K in (2.6), we arrive by means of (3.9) to the inequality

$$\tilde{W}(\bar{\varepsilon})\leq\tilde{W}_+(\bar{\varepsilon}), \tag{3.14}$$

where

$$\tilde{W}_+(\bar{\varepsilon})=\min_{u_e,u_m\in K'}\ \inf_{p_e,p_m\geq 0}L\left(u_e,u_m,p_e,p_m\right), \tag{3.15}$$

and L is the functional given by

$$L\left(u_e,u_m,p_e,p_m\right)=\int_\Omega\left[u_e(x)p_e(x)+u_m(x)p_m(x)\right]dx-\int_\Omega f_*\left(x;p_e(x),p_m(x)\right)dx. \tag{3.16}$$

We emphasize that for *concave f*, $\tilde{W}_+(\bar{\varepsilon})=\tilde{W}(\bar{\varepsilon})$.

Next, we simplify the expression for $\tilde{W}_+(\bar{\varepsilon})$, by noting that

$$\min_{u_e,u_m\in K'}\ \inf_{p_e,p_m\geq 0}L\left(u_e,u_m,p_e,p_m\right)=\inf_{p_e,p_m\geq 0}\ \inf_{u_e,u_m\in K'}L\left(u_e,u_m,p_e,p_m\right), \tag{3.17}$$

since the order in which infima are evaluated can be interchanged. It follows that

$$\tilde{W}_+(\bar{\varepsilon})=\inf_{p_e(x),p_m(x)\geq 0}\left\{\tilde{W}_o(\bar{\varepsilon})-\int_\Omega f_*\left(x;p_e(x),p_m(x)\right)dx\right\}, \tag{3.18}$$

where

$$\tilde{W}_o(\bar{\boldsymbol{\varepsilon}}) = \inf_{u_e, u_m \in K'} \left\{ \int_\Omega \left[u_e(\mathbf{x}) p_e(\mathbf{x}) + u_m(\mathbf{x}) p_m(\mathbf{x}) \right] dx \right\}. \tag{3.19}$$

By introducing the new variables $\mu_o = \frac{3}{2} p_e$ and $\kappa_o = \frac{2}{9} p_m$, and expressing u_e and u_m in terms of ε_e and ε_m, respectively, we are able to rewrite the integrand in (3.19) in terms of the quadratic energy function

$$w_o(\mathbf{x}, \boldsymbol{\varepsilon}) = \phi_o(\mathbf{x}; \varepsilon_e, \varepsilon_m) = \frac{2}{3} \mu_o(\mathbf{x}) \varepsilon_e^2 + \frac{9}{2} \kappa_o(\mathbf{x}) \varepsilon_m^2, \tag{3.20}$$

which corresponds to a linear-elastic, isotropic, heterogeneous solid with nonnegative, but otherwise arbitrary, shear modulus $\mu_o(\mathbf{x})$, and bulk modulus $\kappa_o(\mathbf{x})$. With this notation, the new variational principle takes the final form

$$\tilde{W}_+(\bar{\boldsymbol{\varepsilon}}) = \inf_{\mu_o(\mathbf{x}), \kappa_o(\mathbf{x}) \geq 0} \left\{ \tilde{W}_o(\bar{\boldsymbol{\varepsilon}}) + V(\mu_o, \kappa_o) \right\}, \tag{3.21}$$

where

$$\tilde{W}_o(\bar{\boldsymbol{\varepsilon}}) = \min_{\boldsymbol{\varepsilon} \in K(\bar{\boldsymbol{\varepsilon}})} \int_\Omega w_o(\mathbf{x}, \boldsymbol{\varepsilon}(\mathbf{x}))\, dx, \tag{3.22}$$

and where V is the functional specified by the function $v(\mathbf{x}; \mu_o, \kappa_o) = -f_*\left(\mathbf{x}; \frac{2}{3}\mu_o, \frac{9}{2}\kappa_o \right)$, such that

$$V(\mu_o, \kappa_o) = \int_\Omega v\left(\mathbf{x}; \mu_o(\mathbf{x}), \kappa_o(\mathbf{x})\right) dx. \tag{3.23}$$

We emphasize once again that, under the hypothesis of concave f, which is usually valid in plasticity, the new variational principle and the classical principle of minimum potential energy are completely equivalent, and we have equality in (3.14). Thus, under these circumstances, the new variational principle can be given the following interpretation: the effective behavior of the nonlinear solid, as characterized by its effective energy (2.4), can be determined alternatively in terms of the effective energy of a linear, heterogeneous comparison solid, with variable (within each phase) moduli, $\mu_o(\mathbf{x})$ and $\kappa_o(\mathbf{x})$, whose precise variation is determined by the new

variational principle (3.21). Clearly, the new variational principle is in general at least as difficult to evaluate as the original classical variational principle for the nonlinear solid. The main advantage of the new variational principle is that it allows useful approximations that cannot be obtained directly from the classical variational principle. One possible approximation in (3.21) is to replace the infimum over the set of (arbitrarily) variable, nonnegative moduli by the smaller set of piecewise constant, nonnegative moduli. We will consider the application of this approximation in the determination of bounds for the effective properties of nonlinear composites in the following section. This type of approximation was introduced directly by PONTE CASTAÑEDA (1991a). In particular, it was found in that work that when this approximation was used in conjunction with the well-known Hashin-Shtrikman bounds for linear composites, nonlinear bounds resulted that were in agreement (see also WILLIS, 1991) with earlier bounds determined earlier by PONTE CASTAÑEDA and WILLIS (1988) using a nonlinear-extension of the Hashin-Shtrikman variational principles due to WILLIS (1983) and TALBOT and WILLIS (1985). We note that the new variational principle is restricted [for equality to hold in (3.14)] thus far to isotropic phases, although it can be used in conjunction with anisotropic microgeometries leading to anisotropic effective properties. On the other hand, the Talbot-Willis variational principle can be used to obtain Hashin-Shtrikman bounds for composites with anisotropic phases, although some level of approximation is required for computational convenience (see DENDIEVEL, BONNET and WILLIS, 1991).

Finally, we remark that statement (3.9) can be rewritten in the form

$$w(\mathbf{x};\boldsymbol{\varepsilon}) \le \inf_{\mu_o,\kappa_o \ge 0} \left\{ w_o(\mathbf{x},\boldsymbol{\varepsilon}) - v(\mathbf{x};\mu_o,\kappa_o) \right\}, \tag{3.24}$$

(with equality if w is such that f is concave), where $v(\mathbf{x};\mu_o,\kappa_o)$ takes the corresponding form

$$v(\mathbf{x};\mu_o,\kappa_o) = \sup_{\boldsymbol{\varepsilon}} \left\{ w(\mathbf{x},\boldsymbol{\varepsilon}) - w_o(\mathbf{x},\boldsymbol{\varepsilon}) \right\}. \tag{3.25}$$

With this representation of the local energy functions, the new variational principle is seen to be nothing more that the global version of the relation (3.24). Although we have already remarked that the new variational principle (3.21) is only exact for isotropic phases, provided that f is concave, the form (3.25) for $v(\mathbf{x};\mu_o,\kappa_o)$ suggests that the new principle could also be used for composites

with anisotropic phases. Indeed this is the case, but in this event, equality could not hold in (3.24), and the new variational principle in the given form, would yield only an upper bound approximation to the effective behavior of the nonlinear heterogeneous solid $\bar{W}$, that would be expected to get progressively weaker with the degree of anisotropy in the constituent phases. Improved bounds could be obtained by the introduction of anisotropic comparison materials. This is outside the scope of this paper, and will be pursued elsewhere.

3.2. *Minimum complementary energy formulation*

Given the function ψ defined by (3.5), we make use of the same change of variables h as in the previous subsection to define a function $g:\Omega\times R^+\times R^+\to R^+$ such that

$$g(\mathbf{x};v_e,v_m)=\psi\big(\mathbf{x};h^{-1}(v_e),h^{-1}(v_m)\big). \tag{3.26}$$

By the properties of ψ, enunciated at the beginning of this section, g is a nonnegative function with $g(\mathbf{x};0,0)=0$. Further, g is continuous, but not necessarily convex, in its last two arguments. However, it is shown in Appendix II that if f is concave, then g is convex. As mentioned earlier, this is the usual situation in plasticity. For instance, for the same example of an incompressible power-law material given in the previous subsection, $u\sim\sigma_e^{n+1}$ $(n\geq 1)$, and $g\sim v_e^{\frac{n+1}{2}}$, which are both convex functions.

We can then define (see Appendix I) the *convex* polar function $g^*:\Omega\times R^+\times R^+\to R^+$ via

$$g^*(\mathbf{x};q_e,q_m)=\sup_{v_e,v_m\geq 0}\big\{v_eq_e+v_mq_m-g(\mathbf{x};v_e,v_m)\big\}, \tag{3.27}$$

so that

$$g(\mathbf{x};v_e,v_m)\geq\sup_{q_e,q_m\geq 0}\big\{v_eq_e+v_mq_m-g^*(\mathbf{x};q_e,q_m)\big\}, \tag{3.28}$$

with *equality* if g is convex.

Introducing the functional

$$G(v_e,v_m)=\int_\Omega g\big(\mathbf{x};v_e(\mathbf{x}),v_m(\mathbf{x})\big)\,dx, \tag{3.29}$$

and following a procedure analogous to that of the previous subsection, we arrive at the following inequality involving the effective complementary energy of the nonlinear heterogeneous solid as characterized by (2.11), namely,

$$\tilde{U}(\overline{\sigma}) \geq \tilde{U}_{-}(\overline{\sigma}), \tag{3.30}$$

where

$$\tilde{U}_{-}(\overline{\sigma}) = \min_{v_e, v_m \in S'} \sup_{q_e, q_m \geq 0} M(v_e, v_m, q_e, q_m), \tag{3.31}$$

M is the "saddle" functional given by

$$M(v_e, v_m, q_e, q_m) = \int_{\Omega} [v_e(\mathbf{x}) q_e(\mathbf{x}) + v_m(\mathbf{x}) q_m(\mathbf{x})]\, dx - \int_{\Omega} g^*(\mathbf{x}; q_e(\mathbf{x}), q_m(\mathbf{x}))\, dx, \tag{3.32}$$

and S' is the set induced by the above change of variables on the set of admissible stresses S. We emphasize that for concave f (and therefore convex g), $\tilde{U}_{-}(\overline{\sigma}) = \tilde{U}(\overline{\sigma})$.

In order to simplify the above expression for $\tilde{U}_{-}$, we are required to interchange the order of the minimum and the supremum. This is allowed by Prop. VI.2.3 of EKELAND and TEMAM (1974), because M is affine and therefore convex in v_e, v_m, and concave (because g^* is convex) in q_e, q_m, and $\lim_{v_e, v_m \to \infty} M(v_e, v_m, q_e^{(o)}, q_m^{(o)}) = \infty$ for $q_e^{(o)}$, $q_m^{(o)} > 0$ fixed. Thus, we conclude that

$$\min_{v_e, v_m \in S'} \sup_{q_e, q_m \geq 0} M(v_e, v_m, q_e, q_m) = \sup_{q_e, q_m \geq 0} \inf_{v_e, v_m \in S'} M(v_e, v_m, q_e, q_m). \tag{3.33}$$

This leads to the following restatement of (3.31)

$$\tilde{U}_{-}(\overline{\sigma}) = \sup_{\mu_o(\mathbf{x}), \kappa_o(\mathbf{x}) \geq 0} \{\tilde{U}_o(\overline{\sigma}) - V(\mu_o, \kappa_o)\}, \tag{3.34}$$

where

$$\tilde{U}_o(\overline{\sigma}) = \min_{\sigma \in S(\overline{\sigma})} \int_{\Omega} u_o(\mathbf{x}, \sigma(\mathbf{x}))\, dx \tag{3.35}$$

is the effective complementary energy of the comparison composite, with $u_o = w_o^*$ [see (3.20)], and where the functional V is given by relation (3.23), with $v(\mathbf{x}; \mu_o, \kappa_o) = g^*\left(\mathbf{x}; \frac{1}{6\mu_o}, \frac{1}{2\kappa_o}\right)$. This last expression for v can be expressed directly in terms of u and u_o via

$$v(\mathbf{x};\mu_o,\kappa_o)=\sup_{\boldsymbol{\sigma}}\{u_o(\mathbf{x},\boldsymbol{\sigma})-u(\mathbf{x},\boldsymbol{\sigma})\}. \tag{3.36}$$

In terms of this representation, it is easy to verify that the v functions given by (3.25) and (3.36) are in fact identical (and hence the choice of the same names). Simply note that from (3.36), and the fact that the order of suprema can be interchanged, it follows that

$$\begin{aligned} v(\mathbf{x};\mu_o,\kappa_o) &= \sup_{\boldsymbol{\sigma}}\left\{\sup_{\boldsymbol{\varepsilon}}\{\boldsymbol{\varepsilon}\cdot\boldsymbol{\sigma}-w_o(\mathbf{x},\boldsymbol{\varepsilon})\}-u(\mathbf{x},\boldsymbol{\sigma})\right\} \\ &= \sup_{\boldsymbol{\varepsilon}}\left\{\sup_{\boldsymbol{\sigma}}\{\boldsymbol{\varepsilon}\cdot\boldsymbol{\sigma}-u(\mathbf{x},\boldsymbol{\sigma})\}-w_o(\mathbf{x},\boldsymbol{\varepsilon})\right\} \\ &= \sup_{\boldsymbol{\varepsilon}}\{w(\mathbf{x},\boldsymbol{\varepsilon})-w_o(\mathbf{x},\boldsymbol{\varepsilon})\}, \end{aligned} \tag{3.37}$$

which is in agreement with (3.25). This result can also be obtained by means of result (II.7) in Appendix II, which shows that $f_*(p_e,p_m)=-g^*\left(\frac{1}{9p_e},\frac{9}{4p_m}\right)$. Also, note the following counterpart to (3.24)

$$u(\mathbf{x},\boldsymbol{\sigma})\geq \sup_{\mu_o,\kappa_o\geq 0}\{u_o(\mathbf{x},\boldsymbol{\sigma})-v(\mathbf{x};\mu_o,\kappa_o)\}, \tag{3.38}$$

which is the local version of (3.34).

Finally, we study the relation between the two variational principles $\tilde{W}_+$ and $\tilde{U}_-$, as given by (3.21) and (3.34). We begin by evaluating the polar of $\tilde{W}_+$, yielding

$$\begin{aligned} \tilde{W}_+^*(\overline{\boldsymbol{\sigma}}) &= \sup_{\overline{\boldsymbol{\varepsilon}}}\left\{\overline{\boldsymbol{\varepsilon}}\cdot\overline{\boldsymbol{\sigma}}-\tilde{W}_+(\overline{\boldsymbol{\varepsilon}})\right\} \\ &= \sup_{\overline{\boldsymbol{\varepsilon}}}\left\{\overline{\boldsymbol{\varepsilon}}\cdot\overline{\boldsymbol{\sigma}}-\inf_{\mu_o(\mathbf{x}),\kappa_o(\mathbf{x})\geq 0}\left\{\tilde{W}_o(\overline{\boldsymbol{\varepsilon}})+V(\mu_o,\kappa_o)\right\}\right\} \\ &= \sup_{\mu_o(\mathbf{x}),\kappa_o(\mathbf{x})\geq 0}\left\{\sup_{\overline{\boldsymbol{\varepsilon}}}\left\{\overline{\boldsymbol{\varepsilon}}\cdot\overline{\boldsymbol{\sigma}}-\tilde{W}_o(\overline{\boldsymbol{\varepsilon}})\right\}-V(\mu_o,\kappa_o)\right\} \\ &= \sup_{\mu_o(\mathbf{x}),\kappa_o(\mathbf{x})\geq 0}\left\{\tilde{W}_o^*(\overline{\boldsymbol{\sigma}})-V(\mu_o,\kappa_o)\right\}, \end{aligned} \tag{3.39}$$

where once again we have used the fact that the order of the suprema can be interchanged. Next, by specializing the dual version of result (2.15) to the linear comparison composite (*i.e.*, $\tilde{W}_o^*\leq\tilde{U}_o$), we conclude that

$$\tilde{W}_+^*(\overline{\boldsymbol{\sigma}})\leq\tilde{U}_-(\overline{\boldsymbol{\sigma}}). \tag{3.40}$$

For a *composite* (see the discussion following (2.15)), $\tilde{W}_o^* = \tilde{U}_o$, and therefore we have $\tilde{W}_+^* = \tilde{U}_-$ in place of (3.40), which demonstrates that there is no *duality gap* between the minimum potential and minimum complementary energy versions of the new variational principle. Further, if f is concave, then we have equality in (3.14) and (3.30), simultaneously, and therefore we also have equality between $\tilde{W}^*$ and $\tilde{U}$, which is in agreement (as it should) with the discussion following (2.15). Hence, in the application of the new variational principle, one may utilize either one of the two versions of the variational principle. The choice of the version will usually be dictated by convenience.

4. Application to Bounds

In this section, we are concerned with the determination of bounds for the effective properties of composite materials in the context of plasticity. The approach will be to make use of well-known bounds for linear-elastic composites in the new variational principles to induce corresponding bounds on the effective properties of nonlinear plastic composites. Although the procedure can be applied more generally than we will consider in this section, we provide a sampling of the possible applications of the method by including a new derivation of the classical Voigt/Reuss bounds for generally anisotropic composites, new forms for the Hashin-Shtrikman bounds for isotropic and transversely isotropic composites, and completely new higher-order bounds of the Beran type also for isotropic composites. We will be interested only in materials with energy functions satisfying the concavity of f hypothesis, which is quite adequate in plasticity. In this event, we have complete equivalence between the new and classical variational principles, and it is then clear from either (3.21) or (3.34) that bounds on the effective energy-density functions of the linear comparison composites can be utilized, at least in principle, to yield corresponding bounds on the effective energy-density functions of nonlinear composites.

In particular, we will study heterogeneous solids with n distinct phases, each of which is homogeneous and isotropic. The volume fractions of each phase will be assumed to be given.

According to relation (3.1), we can characterize the energy-density functions of the different phases in terms of functions $\phi^{(r)}:\mathrm{R}^+ \times \mathrm{R}^+ \to \mathrm{R}^+$ ($r = 1, \ldots, n$), such that the local energy-density function of the composite is given by

$$\phi(\mathbf{x};\varepsilon_e,\varepsilon_m) = \sum_{r=1}^{n} \chi^{(r)}(\mathbf{x})\phi^{(r)}(\varepsilon_e,\varepsilon_m), \tag{4.1}$$

where $\chi^{(r)}(\mathbf{x})$ is the characteristic function of phase r (this function vanishes, unless $\mathbf{x}$ is in phase r, in which case it equals unity). However, because of the usual representation in plasticity of the total strain as the sum of the (linear) elastic strain and the (nonlinear) plastic strain, we find it more convenient to represent the constitutive relation for the composite in terms of the corresponding complementary energy-density functions of each phase, $\psi^{(r)}$, with a representation for the complementary energy-density function of the composite ψ completely analogous to (4.1). We will restrict our attention to the class of complementary-energy functions

$$\psi^{(r)}(\sigma_e,\sigma_m) = \varphi^{(r)}(\sigma_e) + \frac{1}{2\kappa^{(r)}}\sigma_m^2, \tag{4.2}$$

where the $\varphi^{(r)}$ are convex, nonnegative functions (and such that the corresponding f functions are concave), and $\kappa^{(r)}$ are nonnegative given constants. The form (4.2) is quite appropriate in plasticity since the plastic strain is usually taken to be traceless (incompressible), so that the compressible part of the constitutive relation is due only to the elastic strain, and is hence linear (corresponding to a quadratic potential). Thus, $\kappa^{(r)}$ is the (elastic) bulk modulus of phase r, and $\varphi^{(r)}$ characterizes the total shear behavior of phase r (both elastic and plastic contributions). For instance, for a linear-elastic material,

$$\varphi^{(r)}(\sigma_e) = \frac{1}{6\mu^{(r)}}\sigma_e^2, \tag{4.3}$$

where $\mu^{(r)} \geq 0$ is the elastic shear modulus of the material. More generally, a plastic potential will usually be added to the potential in (4.3) to account for the plastic strains. In our development in the following sections, we will not specialize our consideration to any specific form for the plastic potential, and our results will hold for arbitrary energy-density functions of the form (4.2), with

the given hypotheses on $\varphi^{(r)}$ and $\kappa^{(r)}$. We further note that the volume fractions of each phase are given by

$$c^{(r)} = \int_{\Omega} \chi^{(r)}(\mathbf{x})\, dx \tag{4.4}$$

and are such that $\sum_{r=1}^{n} c^{(r)} = 1$.

In accord with the above-mentioned representation for the local complementary energy-density function $u(\mathbf{x},\boldsymbol{\sigma}) = \psi(\mathbf{x};\sigma_e,\sigma_m)$, we will make use of the complementary energy formulation (3.34) of the new variational principle, although equivalent results would be obtained by means of the dual potential energy formulation (3.21). Thus, we will consider first in § 4.1 lower bounds for the effective energy function $\tilde{U}$, and then continue in §4.2 with the harder problem of determining upper bounds for $\tilde{U}$. The relationship of the bounds to actual microstructures, and concomitant implications for optimality of the bounds will be addressed briefly in § 4.3.

4.1. *Lower bounds*

As indicated in § 3, lower bounds for the effective complementary energy of the nonlinear composite $\tilde{U}$ can be obtained directly from (3.34) by restricting the class of completely arbitrary (nonnegative) comparison moduli to the smaller class of piecewise constant (nonnegative) moduli. Thus, in the case of composites characterized by (4.2), we consider a comparison composite with shear moduli given by

$$\mu_o(\mathbf{x}) = \sum_{r=1}^{n} \chi^{(r)}(\mathbf{x})\mu_o^{(r)}, \tag{4.5}$$

where the $\mu_o^{(r)}$ are nonnegative constants. We note that we have selected the same microstructural distribution for the comparison composite as for the nonlinear composite. Also, we remark that because the "compressive" terms in the $\psi^{(r)}$ functions are quadratic, the supremum problem over the comparison bulk moduli in (3.34) is solved exactly by

$$\kappa_o(\mathbf{x}) = \sum_{r=1}^{n} \chi^{(r)}(\mathbf{x}) \kappa_o^{(r)}, \tag{4.6}$$

where $\kappa_o^{(r)} = \kappa^{(r)}$.

It follows from this discussion that

$$\tilde{U}(\overline{\sigma}) \geq \sup_{\mu_o^{(r)} \geq 0} \left\{ \tilde{U}_o(\overline{\sigma}) - \sum_{r=1}^{n} c^{(r)} v^{(r)}(\mu_o^{(r)}) \right\}, \tag{4.7}$$

where $\tilde{U}_o$ corresponds to the effective complementary energy of the linear comparison composite with shear and bulk moduli characterized by (4.5) and (4.6), respectively, and where the functions $v^{(r)}$ take the simplified form

$$v^{(r)}(\mu_o^{(r)}) = \sup_{s \geq 0} \left\{ \frac{1}{6\mu_o^{(r)}} s^2 - \varphi^{(r)}(s) \right\}. \tag{4.8}$$

We note for later reference that

$$\varphi^{(r)}(s) = \sup_{\mu_o^{(r)} \geq 0} \left\{ \frac{1}{6\mu_o^{(r)}} s^2 - v^{(r)}(\mu_o^{(r)}) \right\}. \tag{4.9}$$

Thus, any lower bound for $\tilde{U}_o$ can be utilized in (4.7) to induce a lower bound on $\tilde{U}$. We note that for nonlinear composites, just as for linear composites, the bounds on the effective energy will depend on the prescribed microstructural information. For example, the Hashin-Shtrikman bounds for isotropic composites will be tighter than the Voigt/Reuss bounds for generally anisotropic composites. However, it is important to note that the characterization of microstructure is different for linear and nonlinear materials. For example, a periodic linear composite with unit cell possessing icosahedral symmetry leads to isotropic effective behavior for the composite. The same is *not* true in general of a periodic nonlinear composite with an icosahedral unit cell. This is because linear behaviors can be characterized by (finite dimensional) tensors, whereas nonlinear behaviors require in general characterization in terms of (infinite dimensional) functions. Notwithstanding this difficulty, we observe that the class of isotropic linear *microstructures* includes the class of isotropic nonlinear microstructures. Hence, nonlinear bounds based on linear

isotropic microstructures will also hold for nonlinear isotropic microstructures. Analogous results would hold for any other material symmetries for the composite (*e.g.* transversely isotropic). Thus, a nonlinear lower bound for nonlinear composites with a given symmetry results by replacing $\tilde{U}_o$ in (4.7) by a lower bound for $\tilde{U}_o$ with the same symmetry.

4.1.1. *Voigt bound.* We begin by considering the case of a generally *anisotropic* composite with phase potentials (4.2) in given volume fractions $c^{(r)}$. Thus, in this case, we require a lower bound for the effective complementary-energy function of a linear comparison composite with shear and bulk moduli characterized by (4.5) and (4.6), in volume fractions $c^{(r)}$. Such a bound is provided by the so-called Voigt estimate

$$\tilde{U}_o^{(V)}(\overline{\boldsymbol{\sigma}}) = \frac{1}{6\mu_o^{(V)}}\overline{\sigma}_e^2 + \frac{1}{2\kappa_o^{(V)}}\overline{\sigma}_m^2, \tag{4.10}$$

where $\mu_o^{(V)} = \sum_{r=1}^{n} c^{(r)}\mu_o^{(r)}$ and $\kappa_o^{(V)} = \sum_{r=1}^{n} c^{(r)}\kappa_o^{(r)}$. This result can be readily obtained from the principle of minimum complementary energy (HILL, 1952; PAUL, 1960). Thus, use of this lower bound for the linear comparison composite into (4.7) leads to the following lower bound for the effective energy of the nonlinear composite

$$\tilde{U}^{(V)}(\overline{\boldsymbol{\sigma}}) = \sup_{\mu_o^{(r)} \geq 0}\left\{\frac{1}{6\mu_o^{(V)}}\overline{\sigma}_e^2 - \sum_{r=1}^{n} c^{(r)}\nu^{(r)}(\mu_o^{(r)})\right\} + \frac{1}{2\kappa^{(V)}}\overline{\sigma}_m^2, \tag{4.11}$$

where $\kappa^{(V)} = \sum_{r=1}^{n} c^{(r)}\kappa^{(r)}$. The above bound involves $2n$ optimizations (two for each phase), but it can be greatly simplified by means of the identity of Appendix III

$$\frac{1}{\mu_o^{(V)}} = \inf_{\substack{\omega^{(r)} \\ \overline{\omega}=0}}\left\{\sum_{r=1}^{n} \frac{c^{(r)}}{\mu_o^{(r)}}\left(1-\omega^{(r)}\right)^2\right\}, \tag{4.12}$$

where the infimum is over the set of variables $\omega^{(r)}$ ($r = 1,\ldots, n$), which are subject to a zero-average constraint $\overline{\omega} = \sum_{r=1}^{n} c^{(r)}\omega^{(r)} = 0$. With this identity, the expression of the bound in (4.11) leads to

$$\tilde{U}^{(V)}(\overline{\boldsymbol{\sigma}}) = \sup_{\mu_o^{(r)}\geq 0}\left\{\inf_{\substack{\omega^{(r)}\\ \overline{\omega}=0}}\left\{\sum_{r=1}^{n} c^{(r)}\left[\frac{1}{6\mu_o^{(r)}}\left(1-\omega^{(r)}\right)^2\overline{\sigma}_e^2 - \nu^{(r)}(\mu_o^{(r)})\right]\right\}\right\} + \frac{1}{2\kappa^{(V)}}\overline{\sigma}_m^2. \tag{4.13}$$

Next, recalling that $-\nu^{(r)}$ is concave in $\mu_o^{(r)}$ (since g^* is convex), and noting that the expression inside the square brackets is convex in $\omega^{(r)}$, the Saddle Point Theorem (EKELAND and TEMAM, 1974; § VI.4.2) allows the interchange of the order in which the suprema and infima are evaluated to arrive at

$$\tilde{U}^{(V)}(\overline{\boldsymbol{\sigma}}) = \inf_{\substack{\omega^{(r)}\\ \overline{\omega}=0}}\left\{\sum_{r=1}^{n} c^{(r)}\left[\sup_{\mu_o^{(r)}\geq 0}\left\{\frac{1}{6\mu_o^{(r)}}\left(1-\omega^{(r)}\right)^2\overline{\sigma}_e^2 - \nu^{(r)}(\mu_o^{(r)})\right\}\right]\right\} + \frac{1}{2\kappa^{(V)}}\overline{\sigma}_m^2, \tag{4.14}$$

which reduces finally to

$$\tilde{U}^{(V)}(\overline{\boldsymbol{\sigma}}) = \inf_{\substack{\omega^{(r)}\\ \overline{\omega}=0}}\left\{\sum_{r=1}^{n} c^{(r)}\varphi^{(r)}\left(\left|1-\omega^{(r)}\right|\overline{\sigma}_e\right)\right\} + \frac{1}{2\kappa^{(V)}}\overline{\sigma}_m^2, \tag{4.15}$$

where we have made use of (4.9). To the knowledge of the author, this result is new.

This lower bound for $\tilde{U}$, obtained from the linear Voigt bound via the new variational principle, can be shown to be equivalent to the well-known bound, obtained directly from the principle of minimum complementary energy (for a nonlinear composite) by assuming a constant stress field over the composite,

$$\tilde{U}(\overline{\boldsymbol{\sigma}}) \geq \left(\sum_{r=1}^{n} c^{(r)}\left(\varphi^{(r)}\right)^*\right)^*\left(\overline{\sigma}_e\right) + \frac{1}{2\kappa^{(V)}}\overline{\sigma}_m^2. \tag{4.16}$$

For this reason, we refer to the lower bound (4.15) as the nonlinear Voigt bound. We note that the standard form of the bound (4.16) involves $n + 1$ optimizations (Legendre transforms), whereas

the new form (4.15) involves n optimizations with a simple constraint. For the case of two phases, the new form of the Voigt bound reduces further to

$$\tilde{U}^{(V)}(\overline{\boldsymbol{\sigma}}) = \inf_{\omega}\left\{c^{(1)}\varphi^{(1)}\left(\left|1 - c^{(2)}\omega\right|\overline{\sigma}_e\right) + c^{(2)}\phi^{(2)}\left(\left|1 + c^{(1)}\omega\right|\overline{\sigma}_e\right)\right\} + \frac{1}{2\kappa^{(V)}}\overline{\sigma}_m^2, \tag{4.17}$$

which involves only one optimization.

4.1.2. *Hashin-Shtrikman lower bounds.* Next, we consider nonlinear composites with phases having complementary energy-density functions (4.2), in prescribed volume fractions $c^{(r)}$ distributed in such a way that the overall composite is *isotropic*. In this case, the linear comparison composite to be used in conjunction with (4.7) is taken to be isotropic with shear and bulk moduli, given once again by (4.5) and (4.6), respectively, in volume fractions $c^{(r)}$, and with $\kappa_o^{(r)} = \kappa^{(r)}$. The relevant isotropic, linear lower bound, due to HASHIN and SHTRIKMAN (1963), is provided by the relation

$$\tilde{U}_o^{(HS-)}(\overline{\boldsymbol{\sigma}}) = \frac{1}{6\mu_o^{(HS+)}}\overline{\sigma}_e^2 + \frac{1}{2\kappa_o^{(HS+)}}\overline{\sigma}_m^2, \tag{4.18}$$

where

$$\mu_o^{(HS+)} = \frac{\displaystyle\sum_{r=1}^{n} c^{(r)}\frac{\mu_o^{(r)}}{\mu_o^{(r)} + \hat{\mu}_o^{(+)}}}{\displaystyle\sum_{s=1}^{n} c^{(s)}\frac{1}{\mu_o^{(s)} + \hat{\mu}_o^{(+)}}} \quad \text{and} \quad \kappa_o^{(HS+)} = \frac{\displaystyle\sum_{r=1}^{n} c^{(r)}\frac{\kappa_o^{(r)}}{3\kappa_o^{(r)} + 4\mu_o^{(+)}}}{\displaystyle\sum_{s=1}^{n} c^{(s)}\frac{1}{3\kappa_o^{(r)} + 4\mu_o^{(+)}}}, \tag{4.19}$$

with $\hat{\mu}_o^{(+)} = \dfrac{8\mu_o^{(+)} + 9\kappa_o^{(+)}}{6\left(2\mu_o^{(+)} + \kappa_o^{(+)}\right)}\mu_o^{(+)}$, $\mu_o^{(+)} = \max_r\left\{\mu_o^{(r)}\right\}$, and $\kappa_o^{(+)} = \max_r\left\{\kappa_o^{(r)}\right\}$.

Thus, the nonlinear bound for the class of isotropic composites with potentials (4.2) in prescribed volume fractions $c^{(r)}$ becomes

$$\tilde{U}^{(HS-)}(\overline{\boldsymbol{\sigma}}) = \sup_{\mu_o^{(r)} \geq 0}\left\{\frac{1}{6\mu_o^{(HS+)}}\overline{\sigma}_e^2 + \frac{1}{2\kappa_o^{(HS+)}}\overline{\sigma}_m^2 - \sum_{r=1}^{n} c^{(r)}\nu^{(r)}(\mu_o^{(r)})\right\}, \tag{4.20}$$

where we have replaced $\kappa_o^{(r)}$ by $\kappa^{(r)}$ in the expression $(4.19)_2$ for $\kappa_o^{(HS+)}$. Note that this expression still depends on $\mu_o^{(r)}$. Similarly, we replace $\kappa_o^{(+)}$ by $\kappa^{(+)} = \max_r\{\kappa^{(r)}\}$ in expression $(4.19)_1$ for $\mu_o^{(HS+)}$.

In general, the expression (4.20) for the nonlinear isotropic lower bound is quite complicated. Simplification can be achieved by rewriting the above expression for $\mu_o^{(HS+)}$ in the form

$$\frac{1}{\mu_o^{(HS+)}} = \left[\sum_{r=1}^{n} c^{(r)}\left(\frac{1}{\mu_o^{(r)}} + \frac{1}{\hat{\mu}_o^{(+)}}\right)^{-1}\right]^{-1} - \frac{1}{\hat{\mu}_o^{(+)}}, \tag{4.21}$$

which in turn can be rewritten in the form

$$\frac{1}{\mu_o^{(HS+)}} = \inf_{\substack{\omega^{(r)} \\ \bar{\omega}=0}}\left\{\sum_{r=1}^{n} c^{(r)}\left[\frac{1}{\mu_o^{(r)}}\left(1-\omega^{(r)}\right)^2 + \frac{1}{\hat{\mu}_o^{(+)}}\left(\omega^{(r)}\right)^2\right]\right\}, \tag{4.22}$$

by means of the identity of Appendix III. Similarly, we may write

$$\frac{1}{\kappa_o^{(HS+)}} = \inf_{\substack{\gamma^{(r)} \\ \bar{\gamma}=0}}\left\{\sum_{r=1}^{n} c^{(r)}\left[\frac{1}{\kappa^{(r)}}\left(1-\gamma^{(r)}\right)^2 + \frac{3}{4\mu_o^{(+)}}\left(\gamma^{(r)}\right)^2\right]\right\}, \tag{4.23}$$

where the optimization variables $\gamma^{(r)}$ satisfy the constraint $\bar{\gamma} = \sum_{r=1}^{n} c^{(r)}\gamma^{(r)} = 0$. Therefore, we have that

$$\begin{aligned}\tilde{U}^{(HS-)}(\bar{\sigma}) = \sup_{\mu_o^{(r)} \geq 0}\Bigg\{\inf_{\substack{\omega^{(r)},\bar{\omega}=0 \\ \gamma^{(r)},\bar{\gamma}=0}}\Bigg\{\sum_{r=1}^{n} c^{(r)}\Bigg\{\Bigg[\frac{1}{\mu_o^{(r)}}\left(1-\omega^{(r)}\right)^2 + \frac{1}{\hat{\mu}_o^{(+)}}\left(\omega^{(r)}\right)^2\Bigg]\frac{\bar{\sigma}_e^2}{6} + \ldots \\ \ldots + \left[\frac{1}{\kappa^{(r)}}\left(1-\gamma^{(r)}\right)^2 + \frac{3}{4\mu_o^{(+)}}\left(\gamma^{(r)}\right)^2\right]\frac{\bar{\sigma}_m^2}{2} - v^{(r)}(\mu_o^{(r)})\Bigg\}\Bigg\}\Bigg\}.\end{aligned} \tag{4.24}$$

By the same arguments following (4.13), allowing the interchange of the suprema and infima, we arrive at

$$\tilde{U}^{(HS-)}(\overline{\boldsymbol{\sigma}}) = \inf_{\substack{\omega^{(r)},\overline{\omega}=0 \\ \gamma^{(r)},\overline{\gamma}=0}} \left\{ \sup_{\mu_o^{(r)}\geq 0} \left\{ \sum_{r=1}^{n} c^{(r)} \left\{ \left[\frac{1}{\mu_o^{(r)}} \left(1-\omega^{(r)}\right)^2 + \frac{1}{\hat{\mu}_o^{(+)}} \left(\omega^{(r)}\right)^2 \right] \frac{\overline{\sigma}_e^2}{6} + \ldots \right.\right.\right.$$

(4.25)

$$\left.\left.\left. \ldots + \frac{3}{8\mu_o^{(+)}} \left(\gamma^{(r)}\right)^2 \overline{\sigma}_m^2 - v^{(r)}(\mu_o^{(r)}) + \frac{1}{2\kappa^{(r)}} \left(1-\gamma^{(r)}\right)^2 \overline{\sigma}_m^2 \right\} \right\} \right\}.$$

Making use of the expression for $\hat{\mu}_o^{(+)}$, this can in turn be rewritten

$$\tilde{U}^{(HS-)}(\overline{\boldsymbol{\sigma}}) = \min_s \left\{ \inf_{\substack{\omega^{(r)},\overline{\omega}=0 \\ \gamma^{(r)},\overline{\gamma}=0}} \left\{ \sup_{\mu_o^{(r)}\geq 0} \left\{ \sum_{r=1}^{n} c^{(r)} \left\{ \left[\frac{1}{\mu_o^{(r)}} \left(1-\omega^{(r)}\right)^2 + \frac{1}{\hat{\mu}_o^{(s)}} \left(\omega^{(r)}\right)^2 \right] \frac{\overline{\sigma}_e^2}{6} + \ldots \right.\right.\right.\right.$$

(4.26)

$$\left.\left.\left.\left. \ldots + \frac{3}{8\mu_o^{(s)}} \left(\gamma^{(r)}\right)^2 \overline{\sigma}_m^2 - v^{(r)}(\mu_o^{(r)}) + \frac{1}{2\kappa^{(r)}} \left(1-\gamma^{(r)}\right)^2 \overline{\sigma}_m^2 \right\} \right\} \right\} \right\},$$

where we have used the fact that $\frac{1}{\hat{\mu}_o^{(s)}} = \frac{6}{\mu_o^{(s)}} \left(\frac{2\mu_o^{(s)} + \kappa_o^{(+)}}{\mu_o^{(s)} + 9\kappa_o^{(+)}} \right)$ is a monotonically decreasing function of $\mu_o^{(s)}$. This result finally leads to

$$\tilde{U}^{(HS-)}(\overline{\boldsymbol{\sigma}}) = \min_s \left\{ \inf_{\substack{\omega^{(r)},\overline{\omega}=0 \\ \gamma^{(r)},\overline{\gamma}=0}} \left\{ \sum_{\substack{r=1 \\ r\neq s}}^{n} c^{(r)} \varphi^{(r)}\left(\left|1-\omega^{(r)}\right|\overline{\sigma}_e\right) + c^{(s)}\Delta_o^{(s)} + \left(\sum_{r=1}^{n} \frac{c^{(r)}}{\kappa^{(r)}} \left(1-\gamma^{(r)}\right)^2 \right) \frac{\overline{\sigma}_m^2}{2} \right\} \right\}, \tag{4.27}$$

where

$$\Delta_o^{(s)}\left(\overline{\sigma}_e, \overline{\sigma}_m; \omega^{(t)}, \gamma^{(t)}\right) = \sup_{\mu_o^{(s)}\geq 0} \left\{ \frac{1}{6\mu_o^{(s)}} s^2 - v^{(s)}(\mu_o^{(s)}) \right\}, \tag{4.28}$$

with

$$s = \left[\left(1-\omega^{(s)}\right)^2 \overline{\sigma}_e^2 + \frac{6}{c^{(s)}} \left(\frac{2\mu_o^{(s)} + \kappa_o^{(+)}}{\mu_o^{(s)} + 9\kappa_o^{(+)}} \right) \left(\sum_{t=1}^{n} c^{(t)} \left(\omega^{(t)}\right)^2 \right) \overline{\sigma}_e^2 + \frac{9}{4c^{(s)}} \left(\sum_{t=1}^{n} c^{(t)} \left(\gamma^{(t)}\right)^2 \right) \overline{\sigma}_m^2 \right]^{\frac{1}{2}}. \tag{4.29}$$

Note that s depends on $\mu_o^{(s)}$, and therefore the supremum in (4.28) is not trivially evaluated. Thus, the new expression for the nonlinear Hashin-Shtrikman lower bound involves an $2n$-dimensional constrained optimization as given by (4.27), together with the optimization problem for $\Delta_o^{(s)}$. Also, we note that there are n "branches" to the solution (one for each phase), and the minimum over all

the branches yields the desired lower bound. Additionally, we remark that the expression for the bound simplifies to

$$\tilde{U}^{(HS-)}(\overline{\boldsymbol{\sigma}}) = \min_{s}\left\{\inf_{\substack{\omega^{(r)},\overline{\omega}=0\\ \gamma^{(r)},\overline{\gamma}=0}}\left\{\sum_{\substack{r=1\\ r\neq s}}^{n} c^{(r)}\varphi^{(r)}\left(s^{(r)}\right) + c^{(s)}\varphi^{(s)}\left(s^{(s)}\right) + \left(\sum_{r=1}^{n}\frac{c^{(r)}}{\kappa^{(r)}}\left(1-\gamma^{(r)}\right)^2\right)\frac{\overline{\sigma}_m^2}{2}\right\}\right\}, \tag{4.30}$$

where

$$s^{(r)} = \left|1-\omega^{(r)}\right|\overline{\sigma}_e, \tag{4.31}$$

$$s^{(s)} = \sqrt{\left(1-\omega^{(s)}\right)^2\overline{\sigma}_e^2 + \frac{2}{3c^{(s)}}\left(\sum_{t=1}^{n} c^{(t)}\left(\omega^{(t)}\right)^2\right)\overline{\sigma}_e^2 + \frac{9}{4c^{(s)}}\left(\sum_{t=1}^{n} c^{(t)}\left(\gamma^{(t)}\right)^2\right)\overline{\sigma}_m^2},$$

if at least one phase is incompressible (so that $\kappa_o^{(+)} \to \infty$). If all phases are incompressible, the expression simplifies further to

$$\tilde{U}^{(HS-)}(\overline{\boldsymbol{\sigma}}) = \min_{s}\left\{\inf_{\substack{\omega^{(r)}\\ \overline{\omega}=0}}\left\{\sum_{\substack{r=1\\ r\neq s}}^{n} c^{(r)}\varphi^{(r)}\left(s^{(r)}\right) + c^{(s)}\varphi^{(s)}\left(s^{(s)}\right)\right\}\right\}, \tag{4.32}$$

where the $s^{(r)}$ are the same as above, and

$$s^{(s)} = \sqrt{\left(1-\omega^{(s)}\right)^2\overline{\sigma}_e^2 + \frac{2}{3c^{(s)}}\left(\sum_{t=1}^{n} c^{(t)}\left(\omega^{(t)}\right)^2\right)\overline{\sigma}_e^2}. \tag{4.33}$$

If there are only two phases, the constraints can be accounted for by letting $\omega^{(1)} = c^{(2)}\omega$ and $\omega^{(2)} = -c^{(1)}\omega$ (with similar forms for $\gamma^{(1)}$ and $\gamma^{(2)}$). Then, when both phases are incompressible, we arrive at the simple expression

$$\tilde{U}^{(HS-)}(\overline{\boldsymbol{\sigma}}) = \min\left\{\begin{array}{l}\inf_{\omega}\left\{c^{(1)}\phi^{(1)}\left[\left|1-c^{(2)}\omega\right|\overline{\sigma}_e\right] + c^{(2)}\phi^{(2)}\left[\sqrt{\left(1+c^{(1)}\omega\right)^2 + \frac{2}{3}c^{(1)}\omega^2}\,\overline{\sigma}_e\right]\right\}\\ \inf_{\omega}\left\{c^{(1)}\phi^{(1)}\left[\sqrt{\left(1-c^{(2)}\omega\right)^2 + \frac{2}{3}c^{(2)}\omega^2}\,\overline{\sigma}_e\right] + c^{(2)}\phi^{(2)}\left[\left|1+c^{(1)}\omega\right|\overline{\sigma}_e\right]\right\}\end{array}\right\}. \tag{4.34}$$

This expression generalizes results given by PONTE CASTAÑEDA (1991b) for a dual-phase incompressible composite with a linear and a nonlinear phase. Another simple result is provided by the porous material, when one phase, say phase 2, is void (the other is incompressible). Then, the lower bound (4.30) reduces to (since $\omega^{(2)} = \gamma^{(2)} = 1$)

$$\tilde{U}^{(HS-)}(\overline{\sigma}) = c^{(1)}\varphi^{(1)}\left(\frac{1}{c^{(1)}}\sqrt{\left(1+\frac{2}{3}c^{(2)}\right)\overline{\sigma}_e^2 + \frac{9}{4}c^{(2)}\overline{\sigma}_m^2}\right), \tag{4.35}$$

first given by PONTE CASTAÑEDA (1991a). WILLIS (1991) has also shown explicitly that this particular result can be obtained from the Talbot-Willis variational principle by proper choice of the homogeneous comparison material. A weaker version of this result, using the Talbot-Willis variational principle, was provided by PONTE CASTAÑEDA and WILLIS (1988). There are other simple cases in the context of isotropic nonlinear composites, but in the interest of time, they will be considered elsewhere.

Finally, we remark that the new variational principle can also be used in connection with bounds for linear anisotropic composites with *special* symmetries to obtain corresponding bounds for nonlinear anisotropic composites with given symmetries (the first results given in this section hold for *generally* anisotropic composites). For instance, the case of fibre-reinforced materials is considered by PONTE CASTAÑEDA and DE BOTTON (1991). These authors make use of the corresponding linear bounds due to HILL (1964), and HASHIN (1965) (see also WALPOLE, 1969). The results are more complicated than the results for the isotropic composite (since they have transversely isotropic symmetry), but have essentially the same form. For illustrative purposes, we include the final result for a composite with incompressible phases

$$\tilde{U}^{(HS-)}(\overline{\sigma}) = \min_{s}\left\{\inf_{\substack{\omega^{(r)},\overline{\omega}=0\\ \gamma^{(r)},\overline{\gamma}=0}}\left\{\sum_{\substack{r=1\\ r\neq s}}^{n} c^{(r)}\varphi^{(r)}\left(s^{(r)}\right) + c^{(s)}\varphi^{(s)}\left(s^{(s)}\right)\right\}\right\}, \tag{4.36}$$

with

$$s^{(r)} = \sqrt{\left(1-\omega^{(r)}\right)^2\left(\overline{\tau}_n^2+\overline{\tau}_p^2\right)+\left(1-\gamma^{(r)}\right)^2\left(\overline{\sigma}_n-\overline{\sigma}_p\right)^2},$$

(4.37)

$$s^{(s)} = \sqrt{\left[\left(1-\omega^{(r)}\right)^2+\frac{1}{c^{(s)}}\sum_{t=1}^{n}c^{(t)}\left(\omega^{(t)}\right)^2\right]\left(\overline{\tau}_n^2+\overline{\tau}_p^2\right)+\left(1-\gamma^{(r)}\right)^2\left(\overline{\sigma}_n-\overline{\sigma}_p\right)^2},$$

where $\overline{\sigma}_n = \overline{\sigma}_{ij}n_in_j$, $\overline{\sigma}_p = \frac{1}{2}\overline{\sigma}_{ij}\left(\delta_{ij}-n_in_j\right)$, $\overline{\tau}_n = \sqrt{3\left(\overline{\sigma}_{ij}\overline{\sigma}_{ik}n_jn_k-\overline{\sigma}_n^2\right)}$, and $\overline{\tau}_p = \sqrt{\overline{\sigma}_e^2-\overline{\tau}_n^2-\left(\overline{\sigma}_n-\overline{\sigma}_p\right)^2}$ are the transversely isotropic invariants of the mean stress $\overline{\boldsymbol{\sigma}}$ implied by the fiber orientation vector $\mathbf{n}$. Results in a different form for this case have also been given by TALBOT and WILLIS (1991), using the Talbot-Willis variational method.

4.2.3. *Higher-order lower bounds.* So far, we have given examples of bounds that are either well-known, or can be obtained, at least in principle, by other procedures. Thus, the Voigt nonlinear bound can be obtained more directly from the classical principle of minimum complementary energy, and the nonlinear Hashin-Shtrikman lower bounds can also be obtained by means of the Talbot-Willis nonlinear extension of the Hashin-Shtrikman variational principle. However, the simple forms (4.15) and (4.17) for the Voigt bound, and (4.27) to (4.32) for the Hashin-Shtrikman bounds are new. In this subsection, we discuss the determination of nonlinear bounds that have not been possible to determine by any other method hitherto.

Although many different higher-order bounds are available for linear-elastic composites, we will restrict our consideration to the third-order bounds of BERAN and MOLYNEUX (1965), and MCCOY (1970). The reason is that while higher-order bounds can have quite complex forms involving many additional geometric parameters (other than the volume fractions), MILTON (1982) has given a particularly simple form for the third-order bounds of dual-phase isotropic composites, which involve only two additional geometric parameters. Milton's form for the bounds $\mu_o^{(M+)}$ and $\kappa_o^{(M+)}$ may be expressed by

$$\frac{1}{\mu_o^{(M+)}} = \left[\sum_{r=1}^{2}c^{(r)}\left(\frac{1}{\mu_o^{(r)}}+\frac{1}{\hat{\mu}_o^{(+)}}\right)^{-1}\right]^{-1}-\frac{1}{\hat{\mu}_o^{(+)}},$$

(4.38)

which is identical in form to (4.21), but with $\hat{\mu}_o^{(+)} = \Theta/6$, where Θ is given by a rather complicated expression [see (3.29) in MILTON (1982)] involving $\mu_o^{(r)}$ and $\kappa_o^{(r)}$, as well as the geometric parameters $\eta^{(r)}$ and $\zeta^{(r)}$, and by

$$\frac{1}{\kappa_o^{(M+)}} = \left[\sum_{r=1}^{2} c^{(r)}\left(\frac{1}{\kappa_o^{(r)}} + \frac{1}{\hat{\kappa}_o^{(+)}}\right)^{-1}\right]^{-1} - \frac{1}{\hat{\kappa}_o^{(+)}}, \tag{4.39}$$

with $\hat{\kappa}_o^{(+)} = \frac{4}{3}\sum_{r=1}^{2} \zeta^{(r)}\kappa_o^{(r)}$. Here, the geometric parameters $\eta^{(1)}$ and $\zeta^{(1)}$ lie in the interval [0, 1], and $\eta^{(2)} = 1 - \eta^{(1)}$ and $\zeta^{(2)} = 1 - \zeta^{(1)}$.

Substitution of these expressions for the comparison moduli into (4.7) yields an isotropic lower bound $\tilde{U}^{(M-)}$ for the effective energy of the nonlinear composite involving the phase potentials $\varphi^{(1)}$ and $\varphi^{(2)}$ of the two phases, the corresponding volume fractions $c^{(1)}$ and $c^{(2)}$, and the two additional geometric parameters $\eta^{(1)}$ and $\zeta^{(1)}$. However, due to the complicated form of $\hat{\mu}_o^{(+)}$ in (4.38), no further simplification of the nonlinear bound is available. Fortunately, for the case of two incompressible phases, the expression for $\hat{\mu}_o^{(+)}$ simplifies to

$$\hat{\mu}_o^{(+)} = \frac{3}{2}\sum_{r=1}^{2} \eta^{(r)}\mu_o^{(r)}, \tag{4.40}$$

which depends only on the geometric parameter $\eta^{(1)}$. Then, use of a procedure very similar to the one following (4.21) for the Hashin-Shtrikman lower bound leads to the nonlinear third-order lower bound

$$\begin{aligned}\tilde{U}^{(M-)}(\overline{\sigma}) = \inf_{\omega,\gamma}\Bigg\{ & c^{(1)}\phi^{(1)}\left(\sqrt{\left(1 - c^{(2)}\omega\right)^2 + \frac{2}{3}c^{(2)}\eta^{(1)}\omega^2\left(1 - \eta^{(2)}\gamma\right)^2}\,\overline{\sigma}_e\right) + \ldots \\ & \ldots + c^{(2)}\phi^{(2)}\left(\sqrt{\left(1 + c^{(1)}\omega\right)^2 + \frac{2}{3}c^{(1)}\eta^{(2)}\omega^2\left(1 + \eta^{(1)}\gamma\right)^2}\,\overline{\sigma}_e\right)\Bigg\},\end{aligned} \tag{4.41}$$

where $\eta^{(2)} = 1 - \eta^{(1)}$. We remark that the the corresponding nonlinear Hashin-Shtrikman lower bound (4.34) follows immediately from this ...it by choosing ...er $\eta^{(1)} = 0$, or $\eta^{(1)} = 1$, whichever yields the lowest value (note that the infimum problem over γ disappears in either case). This is completely analogous to the corresponding result for *linear* two-phase incompressible

composites (MILTON, 1982). Finally, we note that a superficially simpler result is obtained by making use of the potential energy formulation instead of the complementary energy formulation, as we have done above, since the infimum over γ does not appear in the corresponding form for $\tilde{W}^{(M+)}$. However, such an expression will in general be harder to evaluate, because in plasticity the form of the energy function of the phases is generally more complicated than the corresponding form for the complementary energy functions, used in (4.41).

4.2. *Upper bounds and estimates*

It is clear that the approximate version of the new variational principle (4.7) will not be helpful in the determination of upper bounds for $\tilde{U}$. Therefore, we must resort to the exact version of the new variational principle (3.34), which makes the determination of upper bounds for $\tilde{U}$ an intrinsically harder problem than the corresponding lower bound problem. In this subsection, we will attempt to go as far as we can with the problem of rigorous upper bounds, and then revert to the simpler problem of finding estimates for the upper bound, or "upper estimates". This is accomplished essentially by making use of the approximation (4.7) in spite of the difficulty just mentioned. In § 4.3, we will provide some justification for making this approximation, and give an interpretation for the "upper estimates" for $\tilde{U}$.

4.2.1. *Reuss bound.* We begin by considering the generally anisotropic composite made up of phase energy functions (4.2) in prescribed volume fractions $c^{(r)}$, without any further restriction of the microstructure. Then, the effective complementary energy function of the linear comparison composite $\tilde{U}_o$ with arbitrarily inhomogeneous moduli $\mu_o(\mathbf{x})$ and $\kappa_o(\mathbf{x})$ can be shown (by means of the principle of minimum complementary energy) to be bounded above by

$$\tilde{U}_o^{(R)}(\bar{\boldsymbol{\sigma}}) = \frac{1}{6\mu_o^{(R)}}\bar{\sigma}_e^2 + \frac{1}{2\kappa_o^{(R)}}\bar{\sigma}_m^2, \tag{4.42}$$

where

$$\mu_o^{(R)} = \left(\int_\Omega \frac{1}{\mu_o(\mathbf{x})}\,dx\right)^{-1} \quad \text{and} \quad \kappa_o^{(R)} = \left(\int_\Omega \frac{1}{\kappa_o(\mathbf{x})}\,dx\right)^{-1}. \tag{4.43}$$

These expressions are the continuous versions of the standard (discrete) Reuss estimates.

Correspondingly, the effective complementary energy function of the nonlinear composite $\tilde{U}$ is bounded above by

$$\tilde{U}^{(R)}(\overline{\boldsymbol{\sigma}}) = \sup_{\mu_o(\mathbf{x})\geq 0}\left\{\frac{1}{6\mu_o^{(R)}}\overline{\sigma}_e^2 - \int_\Omega \nu(\mathbf{x};\mu_o(\mathbf{x}))\,dx\right\} + \frac{1}{2\kappa^{(R)}}\overline{\sigma}_m^2, \tag{4.44}$$

where

$$\nu(\mathbf{x};\mu_o) = \sup_{s\geq 0}\left\{\frac{1}{6\mu_o}s^2 - \varphi(\mathbf{x};s)\right\}, \tag{4.45}$$

and where $\kappa^{(R)} = \left(\sum_{r=1}^{n}\frac{c^{(r)}}{\kappa^{(r)}}\right)^{-1}$, since the hydrostatic strain of the nonlinear composite is taken to be linear. Additionally, we have that

$$\varphi(\mathbf{x};s) = \sup_{\mu_o\geq 0}\left\{\frac{1}{6\mu_o}s^2 - \nu(\mathbf{x};\mu_o)\right\}, \tag{4.46}$$

which allows further simplification of (4.44), since

$$\begin{aligned}\tilde{U}^{(R)}(\overline{\boldsymbol{\sigma}}) &= \sup_{\mu_o(\mathbf{x})\geq 0}\left\{\int_\Omega\left[\frac{1}{6\mu_o(\mathbf{x})}\overline{\sigma}_e^2 - \nu(\mathbf{x};\mu_o(\mathbf{x}))\right]dx\right\} + \frac{1}{2\kappa^{(R)}}\overline{\sigma}_m^2 \\ &= \int_\Omega \sup_{\mu_o(\mathbf{x})\geq 0}\left\{\frac{1}{6\mu_o(\mathbf{x})}\overline{\sigma}_e^2 - \nu(\mathbf{x};\mu_o(\mathbf{x}))\right\}dx + \frac{1}{2\kappa^{(R)}}\overline{\sigma}_m^2 \\ &= \int_\Omega \varphi(\mathbf{x};\overline{\sigma}_e)\,dx + \frac{1}{2\kappa^{(R)}}\overline{\sigma}_m^2,\end{aligned} \tag{4.47}$$

and therefore

$$\tilde{U}^{(R)}(\overline{\boldsymbol{\sigma}}) = \sum_{r=1}^{n} c^{(r)}\varphi^{(r)}(\overline{\sigma}_e) + \frac{1}{2\kappa^{(R)}}\overline{\sigma}_m^2. \tag{4.48}$$

This last result is the Reuss estimate for the nonlinear composite, which could have been obtained from the principle of minimum complementary energy, applied directly to the nonlinear composite.

The present, less direct, derivation of this result via the new variational principle serves to demonstrate that the new method can at least recover the classical bounds exactly. This derivation serves further to illustrate how the new variational principle may be used in other contexts for which upper bounds may not be available via the classical principles.

Interestingly, we remark that the above result for the nonlinear Reuss bound (4.48) could also have been obtained by means of the approximation (4.7), following a procedure analogous to that used in the derivation of the nonlinear Voigt bound (4.15). This suggests that the approximation introduced in (4.7) is exact in this case. We will give some insight in the reasons behind this phenomenon in § 4.3.

4.2.2. *Hashin-Shtrikman upper estimates.* We consider next the upper bound problem for the class of nonlinear composites with phase potentials (4.2) in fixed volume fractions (4.4), distributed in such a way that the effective behavior of the composite is *isotropic*. In this case, however, direct application of the exact version of the new variational principle (3.34) leads to a much harder problem than the one solved above for the Voigt bound for anisotropic geometries. For this reason, we will not pursue this approach here, and resort instead to the use of the approximation introduced in (4.7). Then, upper bounds for the effective energy $\tilde{U}_o$ of the class of isotropic linear comparison composites with prescribed volume fractions lead to *estimates* for the upper bound of the class of isotropic nonlinear composites with prescribed volume fractions. We will refer to these estimates as "upper estimates" since they may be close to the upper bound. (In fact, the corresponding computation for the Reuss upper bound suggests that in some cases these estimates may yield rigorous bounds). Further, in practical applications, we are rarely interested in precise bounds; instead, estimates are often preferred.

The computation of the "upper estimates" follows a procedure completely analogous to the computation of the lower bounds. Thus, the effective energy function of the linear comparison composite $\tilde{U}_o$ is bounded above by

$$\tilde{U}_o^{(HS+)}(\overline{\sigma}) = \frac{1}{6\mu_o^{(HS-)}}\overline{\sigma}_e^2 + \frac{1}{2\kappa_o^{(HS-)}}\overline{\sigma}_m^2, \tag{4.49}$$

where $\mu_o^{(HS-)}$ and $\kappa_o^{(HS-)}$ are given by expressions similar to (4.19), except that $\hat{\mu}_o^{(+)}$, $\mu_o^{(+)}$ and $\kappa_o^{(+)}$ must be replaced by $\hat{\mu}_o^{(-)} = \dfrac{8\mu_o^{(-)} + 9\kappa_o^{(-)}}{6\left(2\mu_o^{(-)} + \kappa_o^{(-)}\right)}\mu_o^{(-)}$, $\mu_o^{(-)} = \min_r\left\{\mu_o^{(r)}\right\}$ and $\kappa_o^{(-)} = \min_r\left\{\kappa_o^{(r)}\right\}$. Then, use of this upper bound for $\tilde{U}_o$ into (4.7) leads to the "upper estimate" $\tilde{U}^{(HS+)}$ for $\tilde{U}$. The final result for $\tilde{U}^{(HS+)}$ is

$$(4.50)\quad \tilde{U}^{(HS+)}(\overline{\boldsymbol{\sigma}}) = \max_s\left\{\inf_{\substack{\omega^{(r)},\,\overline{\omega}=0\\ \gamma^{(r)},\,\overline{\gamma}=0}}\left\{\sum_{\substack{r=1\\ r\neq s}}^{n} c^{(r)}\varphi^{(r)}\left(\left|1-\omega^{(r)}\right|\overline{\sigma}_e\right) + c^{(s)}\Delta_o^{(s)} + \left(\sum_{r=1}^{n}\frac{c^{(r)}}{\kappa^{(r)}}\left(1-\gamma^{(r)}\right)^2\right)\frac{\overline{\sigma}_m^2}{2}\right\}\right\},$$

where $\Delta_o^{(s)}$ is as given by (4.28) and (4.29), but with $\kappa_o^{(+)}$ replaced by $\kappa_o^{(-)}$. Note that, apart from these modifications, the expression (4.50) for $\tilde{U}^{(HS+)}$ differs from the expression (4.27) for $\tilde{U}^{(HS-)}$ only in that the outermost minimum over all phases has been replaced by a maximum.

When all the phases are incompressible (so that $\kappa_o^{(-)} \to \infty$), the above result simplifies further to

$$(4.51)\qquad \tilde{U}^{(HS+)}(\overline{\boldsymbol{\sigma}}) = \max_s\left\{\inf_{\substack{\omega^{(r)}\\ \overline{\omega}=0}}\left\{\sum_{\substack{r=1\\ r\neq s}}^{n} c^{(r)}\varphi^{(r)}\left(s^{(r)}\right) + c^{(s)}\varphi^{(s)}\left(s^{(s)}\right)\right\}\right\},$$

where $s^{(r)}$ and $s^{(s)}$ are given by $(4.31)_1$ and (4.33), respectively. Note that the only difference in this case between the lower bound and the upper estimate is that the outermost minimum in (4.32) is replaced by a maximum in (4.51). Thus, from a computational point of view, the lower bound and upper estimates should be computed together since they involve the same "branches" which must be minimized in one case, and maximized in the other. A special case of the upper estimate (4.51) has been discussed previously by PONTE CASTAÑEDA (1991b) for two-phase composites with a linear and a nonlinear phase. A special case is the rigidly reinforced material, which has

$$\tilde{U}^{(HS+)}(\overline{\boldsymbol{\sigma}}) = c^{(1)}\varphi^{(1)}\left(\frac{\overline{\sigma}_e}{\sqrt{1+\frac{3}{2}c^{(2)}}}\right), \tag{4.52}$$

where phase 2 has been selected as the reinforcement.

Finally, an upper estimate for the incompressible fiber-reinforced composite with transversely isotropic symmetry is given by the same relation (4.36) for the lower bound, together with (4.37), except that the minimum over all phases is replaced by a maximum.

4.2.3. *Higher-order upper estimates*. Higher-order upper estimates for two-phase nonlinear composites are obtained by making use of the lower bounds of Milton for comparison shear moduli with the same distribution of phases as the nonlinear composite into (4.7). These bounds for the shear modulus $\mu_o^{(M-)}$ and bulk modulus $\kappa_o^{(M-)}$ admit the same forms (4.38) and (4.39) as the corresponding upper bounds $\mu_o^{(M+)}$ and $\kappa_o^{(M+)}$, with the difference that $\hat{\mu}_o^{(+)}$ must be replaced by $\hat{\mu}_o^{(-)} = \Xi/6$, where is given by expression (3.29) in MILTON (1982), and $\hat{\kappa}_o^{(+)}$ is to be replaced by $\hat{\kappa}_o^{(-)} = \frac{4}{3}\left(\sum_{r=1}^{2}\frac{\zeta^{(r)}}{\kappa_o^{(r)}}\right)^{-1}$. These results depend also only on the two geometric parameters $\eta^{(1)}$ and $\zeta^{(1)}$, in addition to the phase volume fractions $c^{(1)}$ and $c^{(2)}$.

As was the case for the lower bound $\tilde{U}^{(M-)}$, the expression for the corresponding upper estimate $\tilde{U}^{(M+)}$ is very complicated, except for the case where the two phases are incompressible. Then, the expression for $\hat{\mu}_o^{(-)}$ reduces to

$$\hat{\mu}_o^{(-)} = \frac{3}{2}\left(\sum_{r=1}^{2}\frac{\eta^{(r)}}{\mu_o^{(r)}}\right)^{-1}, \tag{4.53}$$

which depends also only on $\eta^{(1)}$, and we obtain the upper estimate

$$\tilde{U}^{(M+)}(\overline{\sigma}) = \inf_{\omega}\left\{c^{(1)}\phi^{(1)}\left(\sqrt{\left(1-c^{(2)}\omega\right)^2+\frac{2}{3}c^{(2)}\eta^{(1)}\omega^2}\,\overline{\sigma}_e\right)+\ldots\right.$$

(4.41)

$$\left.\ldots+c^{(2)}\phi^{(2)}\left(\sqrt{\left(1+c^{(1)}\omega\right)^2+\frac{2}{3}c^{(1)}\eta^{(2)}\omega^2}\,\overline{\sigma}_e\right)\right\},$$

which involves only one optimization. The corresponding nonlinear Hashin-Shtrikman upper estimate is obtained by choosing either $\eta^{(1)} = 0$, or $\eta^{(1)} = 1$, whichever yields the higher value.

4.3. *Attainability of the bounds*

In this section, we address briefly the question of *optimality* of the nonlinear Voigt/Reuss and Hashin-Shtrikman bounds. By this we mean whether specific microstructures exist, within a given class of composites, which have effective energy functions $\tilde{U}$ attaining the bounds (for the given class of composites). Further, in the case of anisotropic composites, microstructures saturating the bounds must exist for *every* possible loading configuration, in order for the bound to be optimal. If the bound is saturated only for special loading configurations, the bound is said to be *sharp* (but not optimal). We remark that, even in the context of linear-elastic composites, there are still many open questions in connection with the optimality of bounds, and most of the results that are known to date are for two-phase systems. For this reason, we will limit our attention in this section to two-phase nonlinear composites. We further note that in the context of linear elasticity, the knowledge of optimal bounds for the effective energy functions of a given class of composites does not suffice in general to characterize the set of all possible elasticity tensors within the class of composites, also known as the G-closure. However, in some special cases, with sufficient symmetry, optimal bounds for the effective energy function do suffice to characterize the G-closure.

By construction, the nonlinear bounds can only be optimal (sharp), if the corresponding linear bounds are optimal (sharp). The microstructures that are commonly used in the context of linear-elastic composites to prove optimality are *sequentially laminated* materials (see KOHN and MILTON, 1988). These are hierarchical microstructures obtained by layering the result of a previous lamination procedure with one of the original (or a new) phases in a different direction.

Thus, a rank I laminate (Figure 1a) with orientation $\mathbf{n}_I$ is produced by layering the two phases in prescribed proportions, with a certain layer thickness $\delta_I << 1$. A rank II laminate (Figure 1b) is obtained by layering the rank I laminate with one of the initial phases (or with a third phase) in a different direction $\mathbf{n}_{II}$, in given proportions, with layer thickness $\delta_{II} << 1$, such that $\delta_I/\delta_{II} << 1$. Higher rank laminates may be obtained by iterating this procedure. The main advantage of this class of microstructures is that their effective properties can be computed exactly, because the fields are piecewise constant within the composite.

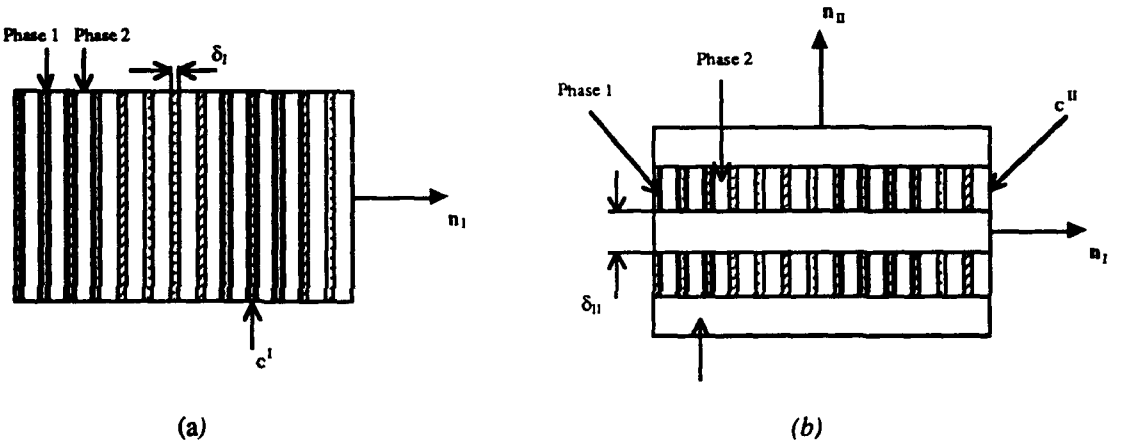

Figure 1. *(a)* A rank-I laminate and *(b)* a rank-II laminate

For the same reason, sequentially laminated microstructures would also appear to be useful in the context of nonlinear composites. Further, the new variational principle seems to be ideally suited for these microstructures also because the fields are piecewise constant, and therefore a piecewise constant choice for the comparison moduli is bound to yield exact results. Thus, we can see that the effective energy $\tilde{U}$ of a two-phase, nonlinear, rank-I laminate (Figure 1a) can be computed exactly via the new variational principle in terms of a two-phase, linear, rank I laminate. However, for a *two*-phase, nonlinear, rank II laminate (Figure 1b), the effective energy of a *three*-phase, linear, rank II laminate is required in order to compute the effective energy of the nonlinear

laminate exactly in terms of the new variational principle. (This is because the fields are different constants within different layers in the laminate, even if they are made of the same material). This observation has the implication that nonlinear bounds that are obtained from optimal (sharp) linear bounds involving sequentially laminated microstructures of rank higher than one are probably not optimal. This is because the microstructures that attain the corresponding linear bounds, from which the nonlinear bounds are derived, cannot be optimal in general for the nonlinear materials. Only in the (unusual) situation where the optimal (sharp) microgeometry is a rank I laminate can we be sure of the corresponding optimality (sharpness) of the nonlinear bound. Additionally, in the special case where the "matrix" phase is linear, the nonlinear lower bound (on $\tilde{U}$) is also optimal, even when the optimal microstructures are laminates of rank higher than one (PONTE CASTAÑEDA, 1991b). In the next subsections, we will address the optimality of the nonlinear Voigt/Reuss bounds, and Hashin-Shtrikman bounds by means of the above-mentioned criteria.

4.3.1. *Voigt/Reuss bounds*. For linear-elastic, two-phase, *anisotropic* composites with prescribed volume fractions, the Voigt/Reuss bounds on the effective energy function $\tilde{U}$ are known to be *non-optimal.* AVELLANEDA (1987) has determined optimal bounds for the effective energy function $\tilde{U}$ for this class of composites. KOHN and LIPTON (1988) have found optimal bounds for the subclass of incompressible two-phase composites, and determined additionally that the Voigt (but not the Reuss) bound is actually optimal in this special case. (Although the new variational principle could in principle be used in conjunction with these optimal bounds to obtain corresponding nonlinear bounds, these optimal linear bounds have rather complicated form, and that is why the simpler Voigt/Reuss bounds were used instead). Thus, the nonlinear Voigt/Reuss bounds are not optimal in general. However, the linear Voigt bound for the two-phase, incompressible composite does involve rank I laminate optimal microstructure, and by the criteria established in the previous paragraph, the nonlinear Voigt bound is optimal for incompressible two-phase composites (but not more generally). Although the nonlinear Reuss bound is not optimal, it is sharp under special loading configurations (see KOHN and LIPTON, 1988 for the corresponding result for linear elastic composites). We add that simple expressions for the effective

properties of nonlinear, rank I laminates have been determined by DE BOTTON and PONTE CASTAÑEDA (1991). Their results are in agreement with the conclusions of this subsection.

4.3.1. *Hashin-Shtrikman lower bound.* For linear-elastic, two-phase, *isotropic* composites with prescribed volume fractions, the Hashin-Shtrikman bounds on the effective energy function $\tilde{U}$ are optimal. This was demonstrated by FRANCFORT and MURAT (1986). However, the corresponding optimal microstructures involve high rank laminates, and therefore these microstructures do not serve to attain the nonlinear Hashin-Shtrikman lower bound on $\tilde{U}$ (as explained above). An exception is provided by the two-phase composite with one linear phase; in this case, the lower bound can be shown to be optimal (PONTE CASTAÑEDA, 1991b). Finally, we note that in general the nonlinear upper estimate for $\tilde{U}$ is probably as far below the optimal upper bound, as the lower bound is below the optimal lower bound (and hence cannot be an upper bound). Thus, if we ignore the rigorous interpretation of the Hashin-Shtriman estimates as bounds, and think of them instead as estimates corresponding to the strongest and weakest composites, then in practice both of them should be held on an equal footing. Thus, it appears that it is only on a *rigorous* sense that the lower bound is "superior" to the upper estimate.

5. Concluding Remarks

In this paper, we have introduced dual versions, (3.21) and (3.34), of a new variational principle that allows the estimation of the effective energy functions of *nonlinear* composites with isotropic phases, as defined by the classical minimum energy and minimum complementary-energy principles, (2.4) and (2.11), respectively. The estimation of the effective properties is in terms of the effective energy functions of suitably optimized *linear* comparison composites. These new variational principles generalize a procedure introduced by PONTE CASTAÑEDA (1991a) for bounding the effective properties of nonlinear composites.

The variational principles have been used to derive bounds and estimates for the effective properties of certain classes of nonlinear composites. Thus, upper and lower bounds of the Voigt

and Reuss type have been derived for the effective complementary-energy function $\tilde{U}$ of generally anisotropic composites, with multiple phases in prescribed volume fractions. Although these bounds are classical, new forms (4.15) and (4.17) have been proposed for the Voigt bound. The nonlinear Voigt/Reuss bounds were found to be in general *not* optimal. Additionally, nonlinear lower bounds of the Hashin-Shtrikman variety have been given for the effective complementary-energy function $\tilde{U}$ of the classes of isotropic, and transversely isotropic, fiber-reinforced composites with prescribed volume fractions. Although bounds of this type are not new, and can be alternatively obtained via the Talbot-Willis variational procedure, the form of the bounds (4.30) to (4.37) given here is new and is the simplest available thus far for general multiple-phase composites. We note in this context that most of the explicit results to date have dealt with porous and rigidly reinforced materials. The Hashin-Shtrikman lower bounds can be shown to be optimal (PONTE CASTAÑEDA, 1991b) for two-phase materials if one phase is linear, but are probably not optimal more generally. We have also proposed that *estimates* obtained by applying the Hashin-Shtrikman upper bounds for the linear comparison composite into the approximate version of the new variational principle should be interpreted as "upper estimates" for the nonlinear composite, corresponding to microstructures exhibiting the least stiff phase as the matrix phase. Finally, the first nonlinear, third-order bounds of the Beran type have also been determined for the class of isotropic, two-phase composites with prescribed volume fractions and given Milton geometric parameters. When both phases are incompressible, remarkably simple forms result, (4.41) for the lower bound and (4.54) for the upper estimate, and they reduces to the corresponding nonlinear Hashin-Shtrikman lower bound and upper estimate for an appropriate choice of the Milton parameters. Clearly, many possibilities exist for further investigation. Central, and probably most difficult, among them is the determination of rigorous *upper* bounds of the Hashin-Shtrikman and higher-order types (for the effective complementary-energy function of nonlinear composites). Also, extensions of the new variational principles are needed to deal with nonlinear composites with anisotropic phases, such as are present in ductile polycrystalline media.

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Appendix I.

The nonnegative functions ϕ and f introduced in § 3 are defined on $\Omega \times R^+ \times R^+$. On the other hand, the usual definition of the polar function refers to functions defined on multiple copies of R (the extended set of reals). In this appendix, we show how to define polars for functions of the type of ϕ and f, defined on multiple copies of R^+. For simplicity, we carry out the development for a one-dimensional function (and drop the dependence on **x**); similar developments hold for multiple-dimensional functions. Also, we consider only the case of the *convex* polar function; analogous results hold for the *concave* polar function. Thus, we consider a function $\tau: R^+ \to R^+$ satisfying the condition $\tau(0) = 0$, and define its extension to R by, say

$$\tilde{\tau}(x) = \tau(|x|), \tag{I.1}$$

where $\tilde{\tau}: R \to R^+$ is also seen to be a nonnegative function satisfying $\tilde{\tau}(0) = 0$.

The convex polar function of $\tilde{\tau}$ is then defined in the usual way by

$$\tilde{\tau}^*(y) = \sup_x \{xy - \tilde{\tau}(x)\}. \tag{I.2}$$

Now, for $y \geq 0$, we have that

$$\begin{aligned} \tilde{\tau}^*(y) &= \sup_{x \geq 0} \{xy - \tilde{\tau}(x)\} \\ &= \sup_{x \geq 0} \{xy - \tau(x)\}, \end{aligned} \tag{I.3}$$

where we have also used the fact that $\tilde{\tau}$ is nonnegative and $\tilde{\tau}(0) = 0$. Note further that $\tilde{\tau}^*$ is also nonnegative, and such that $\tilde{\tau}^*(0) = 0$. Therefore, it makes sense to define $\tau^*: R^+ \to R^+$ via

$$\tau^*(y) = \sup_{x \geq 0} \{xy - \tau(x)\}. \tag{I.4}$$

Finally, it follows from the corresponding duality relation for $\tilde{\tau}$ that

$$\tau(x) \geq \sup_{y \geq 0} \{xy - \tau^*(y)\}, \tag{I.5}$$

with equality if τ is convex.

Appendix II

The main result proved in this appendix is that *concavity* of f implies *convexity* of g. The demonstration of this result has two parts. We begin by recalling some result from § 3. Thus, relation (3.3) can be written (dropping the dependence on x)

$$\psi(\sigma_e,\sigma_m)=\sup_{\varepsilon_e,\varepsilon_m\geq 0}\left\{\frac{2}{3}\sigma_e\varepsilon_e+3\sigma_m\varepsilon_m-\phi(\varepsilon_e,\varepsilon_m)\right\}, \tag{II.1}$$

and rewritten

$$g(v_e,v_m)=\sup_{u_e,u_m\geq 0}\left\{\frac{2}{3}\sqrt{u_e v_e}+3\sqrt{u_m v_m}-f(u_e,u_m)\right\}, \tag{II.2}$$

in terms of f and g. Similarly, relation (3.6) can be written

$$\phi(\varepsilon_e,\varepsilon_m)=\sup_{\sigma_e,\sigma_m\geq 0}\left\{\frac{2}{3}\sigma_e\varepsilon_e+3\sigma_m\varepsilon_m-\psi(\sigma_e,\sigma_m)\right\}, \tag{II.3}$$

which in turn yields

$$f(u_e,u_m)=\sup_{v_e,v_m\geq 0}\left\{\frac{2}{3}\sqrt{u_e v_e}+3\sqrt{u_m v_m}-g(v_e,v_m)\right\}. \tag{II.4}$$

The first part of the demonstration follows from

$$\begin{aligned} f_*(p_e,p_m)&=\inf_{u_e,u_m\geq 0}\left\{u_e p_e+u_m p_m-f(u_e,u_m)\right\}\\ &=\inf_{u_e,u_m\geq 0}\left\{u_e p_e+u_m p_m-\sup_{v_e,v_m\geq 0}\left\{\frac{2}{3}\sqrt{u_e v_e}+3\sqrt{u_m v_m}-g(v_e,v_m)\right\}\right\}\\ &=\inf_{v_e,v_m\geq 0}\left\{\inf_{u_e,u_m\geq 0}\left\{u_e p_e+u_m p_m-\frac{2}{3}\sqrt{u_e v_e}-3\sqrt{u_m v_m}\right\}+g(v_e,v_m)\right\}, \end{aligned} \tag{II.5}$$

where we have used (II.4), the fact that $-\sup\{h\}=\inf\{-h\}$, and interchanged the resulting infimum over the v variables with the infimum over the u variables. But the innermost infimum is trivial to compute, and there results

$$\begin{aligned} f_*(p_e,p_m)&=\inf_{v_e,v_m\geq 0}\left\{-\frac{1}{9}\frac{v_e}{p_e}-\frac{9}{4}\frac{v_m}{p_m}+g(v_e,v_m)\right\}\\ &=-\sup_{v_e,v_m\geq 0}\left\{\frac{1}{9}\frac{v_e}{p_e}+\frac{9}{4}\frac{v_m}{p_m}-g(v_e,v_m)\right\}, \end{aligned} \tag{II.6}$$

which leads to the important identity

$$f_*(p_e,p_m)=-g^*\left(\frac{1}{9p_e},\frac{9}{4p_m}\right). \tag{II.7}$$

The second and final part of the demonstration makes use of (II.2) and of the fact that f is concave (so that $f=f_{**}$) to obtain

$$\begin{aligned} g(v_e,v_m) &= \sup_{u_e,u_m\geq 0}\left\{\frac{2}{3}\sqrt{u_e v_e}+3\sqrt{u_m v_m}-\inf_{p_e,p_m\geq 0}\left\{u_e p_e+u_m p_m-f_*(p_e,p_m)\right\}\right\} \\ &= \sup_{p_e,p_m\geq 0}\left\{\sup_{u_e,u_m\geq 0}\left\{\frac{2}{3}\sqrt{u_e v_e}+3\sqrt{u_m v_m}-u_e p_e-u_m p_m\right\}+f_*(p_e,p_m)\right\}, \end{aligned} \tag{II.8}$$

where we have interchanged the suprema. The innermost supremum is the negative of the innermost infimum in (II.5), and therefore making use of (II.7), we deduce that

$$\begin{aligned} g(v_e,v_m) &= \sup_{p_e,p_m\geq 0}\left\{\frac{1}{9}\frac{v_e}{p_e}+\frac{9}{4}\frac{v_m}{p_m}+f_*(p_e,p_m)\right\} \\ &= \sup_{p_e,p_m\geq 0}\left\{\frac{1}{9}\frac{v_e}{p_e}+\frac{9}{4}\frac{v_m}{p_m}-g^*\left(\frac{1}{9p_e},\frac{9}{4p_m}\right)\right\} \\ &= g^{**}(v_e,v_m). \end{aligned} \tag{II.7}$$

This final result proves that g is convex, and hence completes the proof.

Appendix III

In this appendix, we give a simple proof of the identity (4.12) used throughout the body of the paper. We begin by introducing a Lagrange multiplier to account for the zero-average constraint on the optimization variables, $\omega^{(r)}$. Thus,

$$\text{(III.1)} \qquad \inf_{\substack{\omega^{(r)} \\ \bar{\omega}=0}} \left\{ \sum_{r=1}^{n} \frac{c^{(r)}}{\mu^{(r)}} \left(1-\omega^{(r)}\right)^2 \right\} = \inf_{\omega^{(r)}} \left\{ \sum_{r=1}^{n} \frac{c^{(r)}}{\mu^{(r)}} \left(1-\omega^{(r)}\right)^2 + \sup_{\lambda} \left\{ \lambda \sum_{r=1}^{n} c^{(r)} \omega^{(r)} \right\} \right\},$$

and, by means of the Saddle Point theorem, we have

$$\text{(III.2)} \qquad \inf_{\substack{\omega^{(r)} \\ \bar{\omega}=0}} \left\{ \sum_{r=1}^{n} \frac{c^{(r)}}{\mu^{(r)}} \left(1-\omega^{(r)}\right)^2 \right\} = \sup_{\lambda} \left\{ \inf_{\omega^{(r)}} \left\{ \sum_{r=1}^{n} \frac{c^{(r)}}{\mu^{(r)}} \left(1-\omega^{(r)}\right)^2 + \lambda \sum_{r=1}^{n} c^{(r)} \omega^{(r)} \right\} \right\}.$$

The infimum over the $\omega^{(r)}$ is satisfied by

$$\text{(III.3)} \qquad \omega^{(r)} = 1 - \frac{\lambda}{2} \mu^{(r)},$$

which finally leads to

$$\text{(III.4)} \qquad \begin{aligned} \inf_{\substack{\omega^{(r)} \\ \bar{\omega}=0}} \left\{ \sum_{r=1}^{n} \frac{c^{(r)}}{\mu^{(r)}} \left(1-\omega^{(r)}\right)^2 \right\} &= \sup_{\lambda} \left\{ \sum_{r=1}^{n} c^{(r)} \left\{ \lambda - \frac{\lambda^2}{4} \mu^{(r)} \right\} \right\} \\ &= \sup_{\lambda} \left\{ \lambda - \frac{\lambda^2}{4} \sum_{r=1}^{n} c^{(r)} \mu^{(r)} \right\} \\ &= \left(\sum_{r=1}^{n} c^{(r)} \mu^{(r)} \right)^{-1}. \end{aligned}$$

Reference [12]

In *Anisotropy and Localization of Plastic Deformation*, edited by J.-P. Boehler and A. S. Khan, pp. 15-18. London: Elsevier, 1991.

EFFECTIVE ANISOTROPIC PROPERTIES OF CREEPING COMPOSITES

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ABSTRACT

A recently developed variational principle for estimating the effective properties of nonlinear composites in terms of the corresponding properties of linear composites with the same microstructural distributions of phases is applied to two model anisotropic composite materials. The model materials considered are laminated materials and fibre-reinforced materials, and they correspond in the dilute limit of the inclusion phase to materials reinforced by aligned platelets and fibers, respectively. For simplicity, the power exponent of one of the phases will be taken to be unity corresponding to linear behavior, while the other phase will assumed to satisfy a pure-power creeping law. Both phases will be assumed to be isotropic and incompressible.

INTRODUCTION

A new procedure for estimating the effective properties of composite materials with phases exhibiting nonlinear constitutive behavior has been proposed recently by Ponte Castañeda [6]. The straightforward and versatile procedure expresses the effective properties of the nonlinear composite in terms of the effective properties of a family of linear composites with the same distribution of phases as the nonlinear composite. Thus, bounds and estimates for the effective properties of linear composites can be translated directly into bounds and estimates for the corresponding nonlinear composite. Appropriate references for the linear theory of composites are provided by the works of Christensen [1]. The new procedure was applied in the above reference [6] to composite materials containing a nonlinear isotropic matrix either weakened or reinforced by isotropic distributions of voids or rigid particles, respectively. The case of a ductile matrix reinforced by incompressible elastic particles was studied in [7]. The results of these studies were given in the form of estimates and rigorous bounds for the effective properties of such materials. The Hashin-Shtrikman bounds obtained via the new method directly from the Hashin-Shtrikman [3] bounds for the linear comparison material were found to be an improvement over the corresponding bounds obtained in [8] for the same class of nonlinear materials using the nonlinear extension of the Hashin-Shtrikman variational principle [3] proposed by Talbot and Willis [9]. Recently, however, Willis [11] has shown that the bounds obtained via the new method can also be obtained by the Talbot-Willis method with a better choice of their comparison material. More generally, the new procedure can make use of other bounds and estimates for the linear comparison material to yield corresponding bounds and estimates for nonlinear materials. In fact, the new procedure can be shown [2] to yield exact results for nonlinear composites with special microstructures.

In this paper, we apply the new procedure to composite materials containing a creeping phase of the power-law type and a linearly creeping phase. Although the phases will be assumed to be isotropic, the composite itself will be assumed to be *anisotropic* by identifying

preferred orientations in the microstructural distribution of the phases. Two examples will be considered, both of which have transversely isotropic overall symmetries: laminated materials and fibre-reinforced materials. The first class of materials can be given the interpretation of materials reinforced by aligned platelets for dilute concentrations of the linear phase; whereas, the second class of materials can be given the interpretation of materials reinforced by short fibers for dilute concentrations of the linear phase. For simplicity, we assume that the phases are incompressible and perfectly bonded to each other. In the next three sections, we will give, respectively, a brief definition of effective properties, an introduction to the new variational principle of [6], and application of the principle to the two classes of materials discussed above. More detailed studies including specific results for the nonlinear laminates and fiber-reinforced materials can be found in [2] and [5], respectively.

EFFECTIVE PROPERTIES

Consider a two-phase composite occupying a region of unit volume Ω, such that the local stress potential $U(\boldsymbol{\sigma},\mathbf{x})$ is expressed in terms of the homogeneous phase potentials $U^{(r)}(\boldsymbol{\sigma})$ via

$$U(\boldsymbol{\sigma},\mathbf{x}) = \sum_{r=1}^{2} \chi^{(r)}(\mathbf{x}) U^{(r)}(\boldsymbol{\sigma}), \tag{1}$$

where $\chi^{(r)}$ is the characteristic function of phase r. The phases are assumed to be incompressible and isotropic, so that the potentials $U^{(r)}(\boldsymbol{\sigma})$ can be assumed to depend only on the effective stress $\sigma_e = \sqrt{\frac{3}{2}\mathbf{S}\cdot\mathbf{S}}$, where $\mathbf{S}$ is the deviator of $\boldsymbol{\sigma}$. Thus, we write $U^{(r)}(\boldsymbol{\sigma}) = f^{(r)}(\sigma_e)$, where the $f^{(r)}$ are scalar-valued functions. Then, the local constitutive relation for the creeping material is given by

$$\mathbf{D} = \frac{\partial U}{\partial \boldsymbol{\sigma}}(\boldsymbol{\sigma},\mathbf{x}), \tag{2}$$

where $\mathbf{D}$ is the rate-of-deformation (strain-rate) tensor.

To define the effective properties of the heterogeneous material we introduce, following Hill [4], a uniform constraint boundary condition

$$\boldsymbol{\sigma}\mathbf{n} = \overline{\boldsymbol{\sigma}}\mathbf{n}, \qquad \mathbf{x} \in \partial\Omega, \tag{3}$$

where $\partial\Omega$ denotes the boundary of the composite, $\mathbf{n}$ is its unit outward normal, and $\overline{\boldsymbol{\sigma}}$ is a given constant symmetric tensor. It follows that the average stress is precisely $\overline{\boldsymbol{\sigma}}$, and we *define* the average strain-rate $\overline{\mathbf{D}}$ in a similar manner.

Then,the effective behavior of the composite, or the relation between the average stress and the average strain-rate follows from the principle of minimum complementary energy, which can be stated in the form

$$\tilde{U}(\overline{\boldsymbol{\sigma}}) = \min_{\boldsymbol{\sigma}\in S(\overline{\boldsymbol{\sigma}})} \int_{\Omega} U(\boldsymbol{\sigma},\mathbf{x})\, dV, \tag{4}$$

where $S(\overline{\boldsymbol{\sigma}}) = \{\boldsymbol{\sigma} | \sigma_{ij,j} = 0 \text{ in } \Omega, \text{ and } \sigma_{ij} n_j = \overline{\sigma}_{ij} n_j \text{ on } \partial\Omega\}$ is the set of statically admissible stresses, and where we have assumed convexity of the nonlinear potential $U(\boldsymbol{\sigma},\mathbf{x})$. Thus, assuming that $\tilde{U}(\overline{\boldsymbol{\sigma}})$ is differentiable, we have that

$$\overline{\mathbf{D}} = \frac{\partial \tilde{U}}{\partial \overline{\boldsymbol{\sigma}}}(\overline{\boldsymbol{\sigma}},\mathbf{x}). \tag{5}$$

The task will be to determine bounds and estimates for $\tilde{U}(\overline{\boldsymbol{\sigma}})$, which is known to be convex.

NONLINEAR VARIATIONAL PRINCIPLES

A new variational principle for determining bounds and estimates for the effective properties of nonlinear composites in terms of the effective properties of linear composites was proposed by Ponte Castañeda [6]. In this section, we specialize this result for the case where both phases are incompressible, and phase 2 is linear so that

$$U^{(2)}(\boldsymbol{\sigma}) = \frac{1}{6\mu^{(2)}}\sigma_e^2.$$

The new variational principle is based on a representation of the potential of the nonlinear material in terms of the potentials of a family of linear *comparison* materials. Thus, for a homogeneous nonlinear material with "stronger than quadratic" growth in its potential, $U(\boldsymbol{\sigma})$, and certain additional convexity hypothesis, we have that

$$U(\boldsymbol{\sigma}) = \max_{\mu>0}\{U_o(\boldsymbol{\sigma}) - V(\mu)\}\ , \tag{6}$$

where

$$V(\mu) = \max_{\boldsymbol{\sigma}}\{U_o(\boldsymbol{\sigma}) - U(\boldsymbol{\sigma})\} \tag{7}$$

and where $U_o(\boldsymbol{\sigma})$ is the potential of a linear comparison material with shear modulus μ.

The new variational principle is obtained by making use of relation (6) applied to the nonlinear phase 1 in the complementary energy principle (4) to obtain the following relation for the nonlinear composite

$$\tilde{U}(\overline{\boldsymbol{\sigma}}) = \max_{\mu^{(1)}(\mathbf{x})}\left\{\tilde{U}_o(\overline{\boldsymbol{\sigma}}) - \int_{\Omega^{(1)}} V^{(1)}(\mu^{(1)})\,dV\right\}, \tag{8}$$

where

$$\tilde{U}_o(\overline{\boldsymbol{\sigma}}) = \min_{\boldsymbol{\sigma}\in S(\overline{\boldsymbol{\sigma}})}\int_\Omega U_o(\boldsymbol{\sigma},\mathbf{x})\,dV, \tag{9}$$

is the effective potential of a linear comparison material with local potential $U_o(\boldsymbol{\sigma},\mathbf{x})$ and shear moduli $\mu^{(1)}$ and $\mu^{(2)}$ in phases 1 and 2, respectively. Further, we note that, in general, the comparison moduli $\mu^{(1)}$ are functions of $\mathbf{x}$.

The variational principle described by (8) roughly corresponds to solving a linear problem for a heterogeneous material with arbitrary moduli variation within the nonlinear phase, and then optimizing with respect to the variations in moduli within the nonlinear phase. Thus, the nonlinear material can be thought of as a "linear" material with variable moduli that are determined by prescription (8) in such a way that its properties agree at each $\mathbf{x}$ with those of the nonlinear material. This suggests that if the fields happen to be constant over the nonlinear phase, then the variable moduli $\mu^{(1)}(\mathbf{x})$ can be replaced by constant moduli $\mu^{(1)}$. More generally, however, we have the following lower bound for $\tilde{U}(\overline{\boldsymbol{\sigma}})$, namely,

$$\tilde{U}_{-}(\overline{\boldsymbol{\sigma}}) = \max_{\mu^{(1)}>0}\{\tilde{U}_o(\overline{\boldsymbol{\sigma}}) - c^{(1)}V^{(1)}(\mu^{(1)})\}, \tag{10}$$

where $c^{(1)}$ is the volume fraction of phase 1.

APPLICATION TO LAMINATES

In this section, we summarize from reference [2] the results of applying the new variational principle to a laminated material. For this geometry, it is well known that the stress and strain fields take on different constant values within each phase. This suggests that taking $\mu^{(1)}$ to be constant within the nonlinear phase will lead to an exact result as given by (8) or (10) (in this case $\tilde{U}$ and $\tilde{U}_-$ are identical). The key ingredient in this development is the exact result for the effective potential of the linear laminate. This result can be expressed in the form

$$\tilde{U}(\overline{\boldsymbol{\sigma}}) = \frac{1}{6\hat{\mu}}\tau_n^2 + \frac{1}{6\overline{\mu}}\left[\tau_p^2 + \left(\sigma_n - \sigma_p\right)^2\right], \tag{11}$$

where τ_n, τ_p, σ_n and σ_p are the transversely isotropic invariants of the stress tensor corresponding respectively to the out-of-plane tensile, in-plane hydrostatic, out-of-plane shear and in-plane shear stresses. Also, $\overline{\mu}$ and $\hat{\mu}$ stand, respectively, for the Voigt and Reuss estimates of the shear modulus.

Application of this result into the new variational principle can then be shown to lead to the following simple expression for the effective energy of the laminate

$$\tilde{U}(\overline{\boldsymbol{\sigma}}) = \min_{\omega}\left\{c^{(1)}f^{(1)}\left(s^{(1)}\right) + c^{(2)}f^{(2)}\left(s^{(2)}\right)\right\}, \tag{12}$$

where $s^{(1)}$ and $s^{(2)}$ are functions of ω given by the relations $s^{(1)} = \sqrt{\left(1+c^{(2)}\omega\right)^2\left(\sigma_e^2 - \tau_n^2\right) + \tau_n^2}$ and $s^{(2)} = \sqrt{\left(1-c^{(1)}\omega\right)^2\left(\sigma_e^2 - \tau_n^2\right) + \tau_n^2}$.

Analogous forms can be derived for the fibre-reinforced material, except that for this geometry, the effective energy cannot be given explicitly by (8), but only characterized in the form of bounds by means of (10). Detailed results of these calculations are given in [5], where they are also compared to the results of Talbot and Willis [10] for the same microstructure using the Talbot-Willis variational principle. Finally, both types of results can be averaged over all possible orientations to yield results depicting approximately the effect of inclusion shape on the effective properties of isotropic composites.

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Reference [5]

ON THE DUCTILITY OF LAMINATED MATERIALS

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Abstract

A laminated material is one of the few composite systems for which the effective constitutive behavior can be determined exactly. This is well-known for laminated composites with linear-elastic phases in prescribed volume fractions, for which explicit expressions have been available for their effective moduli for at least thirty years. However, it appears that corresponding expressions for the effective energy functions of laminated composites with phases exhibiting nonlinear constitutive behavior are currently unavailable. While, in principle, the calculation of the effective energy function of a laminated composite is straightforward; in practice, the standard procedure involves the solution of a fairly complicated set of nonlinear algebraic equations. In this paper, we make use of a new variational procedure recently developed by one of the authors to obtain fairly explicit expressions for the effective energy functions of laminated composites with ductile phases in prescribed volume fractions. Explicit results are then computed for ductile/brittle systems, such as an aluminum/alumina laminate, and for a laminated composite made up of two rigid/perfectly plastic phases with different yield stresses. The results—which are representative of other anisotropic geometries, such as fiber-reinforced solids—exhibit a strong anisotropic coupling between different loading modes that is enhanced by material nonlinearity.

1. Introduction

This work is concerned with the determination of the effective behavior of laminated composite materials with plastically deforming phases in prescribed volume fractions. We will deal with the *exact* effective behavior of such materials, and therefore, we will exclude from our consideration the so-called approximate theories of laminated plates (Christensen 1979). In the context of linear elasticity, laminated composites have been studied extensively in the literature. Postma (1955) and White and Angona (1955) concerned themselves with the study of two-phase, periodic laminates in connection with wave propagation in stratified media. Backus (1962) extended their results to multi-phase, non-periodic composites, again in the wave propagation context. The extension to anisotropic layers was considered by Walpole (1969) for aligned, transversely isotropic phases, and by Chou *et al.* (1972) and Pagano (1974) for more general phase anisotropies. Recently, Norris (1991) has developed alternative expressions for the effective moduli tensor of laminated composites with generally anisotropic phases, exploiting the interior and exterior projection tensors of Hill (1972, 1983). Other related works include iterative formulae developed by Francfort and Murat (1986) in the context of linear elasticity allowing simple expressions for the effective moduli of multi-sandwich structures (laminates embedded within laminates of different orientations). These microstructures are of theoretical value in the demonstration of the optimality of bounds for the effective properties of composite materials with more general microstructures (see Kohn 1987, Lipton 1991b). In spite of the large level of activity for linear laminates, briefly summarized above, the theory of nonlinear laminated composites does not seem to have been developed nearly as much. To the knowledge of the authors, the only work thus far in this direction is a generalization of the Francfort-Murat formula for two-phase laminated materials by Kohn (1990) and Milton (1990). Nonlinear results do exist, however, within the context of laminated plate theories.

The justification of the study of nonlinear laminated composites may be partially understood in terms of the following considerations. First, it is a configuration of practical importance: for instance, the use of linear laminated theories in geophysical applications is well-known, but it is also

known that the properties of the materials composing the surface of the Earth may also exhibit nonlinear constitutive behavior, particularly, deep within the surface, where the materials are subject to large compressive stresses. Second, the laminated microstructure corresponds to the simplest possible type of anisotropic composite with nonlinear phases, in the sense that *exact* results may be obtained for its effective properties, as will be shown herein. In this connection, the present work should be considered in the light of a research program attempting to characterize the effective properties of nonlinear composites in general. Thus far, methods have been developed for understanding the effective behavior of nonlinear composites by Talbot and Willis (1985), and by Ponte Castañeda (1991a, c). These methods, although different in essence, can be shown to yield exactly the same results in some cases, but the new method is more general than the first in that it can be used to obtain estimates other than Hashin-Shtrikman bounds and self-consistent estimates (see Willis 1991, and Ponte Castañeda 1991a, c). These methods have been applied to obtain bounds and estimates for the effective properties of nonlinear composites with isotropic overall symmetries by Ponte Castañeda and Willis (1988), and Willis (1989) making use of the Talbot-Willis method, and by Ponte Castañeda (1991a, b, c) making use of the new method. Results for fiber-reinforced composites have also been developed very recently by Talbot and Willis (1991), and by Ponte Castañeda (1991c), and Ponte Castañeda and de Botton (1991). Third, we will find that the laminated microstructure is illustrative of the significant coupling that may arise in nonlinear, anisotropic materials between different loading modes. This nonlinear coupling is also observed in other more complex microstructures, such as fiber-reinforced composites.

The rest of the paper is structured as follows. In § 2, the definition of effective properties is reviewed, and their variational characterization is given in terms of both the classical variational principles, and the new variational principles of Ponte Castañeda (1991a, c). In § 3, the general nonlinear laminated composite is considered, and general formulae for the effective properties are derived in § 4 and 5 for incompressible and compressible laminates, respectively. Finally, in § 6, more specific results are given for two-phase laminates. In particular, the cases of ductile materials reinforced by linear-elastic layers, and of laminates with two perfectly plastic phases are considered.

2. Effective properties and their variational characterization

In this section, we are interested in the characterization of the effective, or overall, constitutive behavior of composites materials with plastically deforming phases. For our purposes, a composite is a heterogeneous material with two distinct length scales: one macroscopic, L, describing the gross size of the specimen and the scale of variation of the applied loading conditions, and a microscopic scale, l, characterizing the size of the typical inhomogeneity, such that $l << L$. More precise definitions can be found in the review article by Kohn (1987).

For simplicity, the constitutive behavior of the phases will be characterized by the deformation theory of plasticity, or equivalently by nonlinear infinitesimal elasticity. However, with minor changes in notation, the results of the analyses of this paper will also be relevant to the high-temperature creeping behavior of composite laminates. Additionally, the results can be used in an *approximate* fashion to suggest yield functions for laminated composites in the context of the incremental theory of plasticity, as suggested by Duva and Hutchinson (1984), and other investigators.

In the following description of effective properties, the composite is assumed to occupy a domain of unit volume Ω, with boundary $\partial\Omega$. Then, the nonlinear plastic behavior of the composite is characterized by means of a complementary-energy density function, $U(\mathbf{x},\boldsymbol{\sigma})$, depending on the position vector $\mathbf{x}$ and the stress field $\boldsymbol{\sigma}(\mathbf{x})$, in such a way that the strain field $\boldsymbol{\varepsilon}(\mathbf{x})$ is given by

$$\boldsymbol{\varepsilon}(\mathbf{x}) = \frac{\partial U(\mathbf{x},\boldsymbol{\sigma})}{\partial \boldsymbol{\sigma}}. \tag{2.1}$$

Following Hill (1963), we define the *effective* constitutive behavior of the heterogeneous solid in terms of the analogous relation

$$\bar{\boldsymbol{\varepsilon}} = \frac{\partial \tilde{U}}{\partial \bar{\boldsymbol{\sigma}}}, \tag{2.2}$$

where $\bar{\boldsymbol{\varepsilon}}$ denotes the mean value of the strain field over Ω, and $\tilde{U}$ refers to the normalized (recall that Ω has unit volume) complementary-energy function of the solid when subjected to the *uniform constraint* boundary condition

$$\boldsymbol{\sigma}\mathbf{n} = \bar{\boldsymbol{\sigma}}\mathbf{n}, \quad \mathbf{x} \in \partial\Omega, \tag{2.3}$$

where $\mathbf{n}$ is the outward unit normal to $\partial\Omega$, and $\bar{\boldsymbol{\sigma}}$ is a constant, symmetric tensor. We recall that under this type of boundary condition, the mean value of the stress over Ω is precisely $\bar{\boldsymbol{\sigma}}$.

The *effective* complementary-energy function of the composite, $\tilde{U}$, can be obtained directly in terms of the principle of minimum complementary energy by means of

$$\tilde{U}(\bar{\boldsymbol{\sigma}}) = \min_{\boldsymbol{\sigma} \in S(\bar{\boldsymbol{\sigma}})} \int_{\Omega} U(\mathbf{x}, \boldsymbol{\sigma}) dv, \tag{2.4}$$

where

$$S(\bar{\boldsymbol{\sigma}}) = \left\{ \boldsymbol{\sigma} \middle| \nabla \cdot \boldsymbol{\sigma} = 0 \text{ in } \Omega, \text{ and } \boldsymbol{\sigma}\mathbf{n} = \bar{\boldsymbol{\sigma}}\mathbf{n} \text{ on } \partial\Omega \right\} \tag{2.5}$$

is the set of statically admissible stress fields. Note that the first set of conditions in (2.5) are the equilibrium equations, and that the minimizing conditions (Euler-Lagrange equations) of (2.4) are the compatibility equations. Further, composite materials typically exhibit sharp interfaces across which the material properties are discontinuous, although the phases are *assumed* to be perfectly bonded. Therefore, across such interfaces, the equilibrium equations must be reinterpreted in terms of continuity of the traction stresses, and correspondingly the compatibility equations must be replaced by continuity of the tangential components of the strain tensor .

We note that, given relation (2.2) in terms of $\tilde{U}$, the problem of characterizing the effective behavior of the composite reduces to that of determining $\tilde{U}$. However, while in principle $\tilde{U}$ can be computed from (2.4); in practice, this variational principle is not very useful for two reasons. First, usually the microstructure of a typical composite is not completely specified; and second, the problem described by (2.4) is a nonlinear one in account of the nonlinear behavior of the constituent phases. For the problem of interest in this paper, the first issue is not a concern because the phase volume-fractions suffice to characterize the microstructure of a laminated composite material.

However, the second issue presents real difficulties. For this reason, we describe next a new variational principle, introduced recently by Ponte Castañeda (1991a, c), which deals precisely with the problem of constitutive nonlinearity. This is accomplished by expressing the effective energy function of the *nonlinear* composite in terms of a variational statement involving the effective energy functions of the class of *linear* comparison composites. Thus, the new variational principle allows the extension of well-known results for linear composites to corresponding results for nonlinear ones. In this paper, we will make use of this variational principle, and of well-known results for the effective properties of linear-elastic laminates, to determine the effective constitutive behavior of ductile laminates. Before proceeding with this task, we briefly review the new variational principle.

The new variational principle for the effective energy of the composite $\tilde{U}$ is obtained by means of the Legendre transformation, applied to a modified set of variables. We will assume that the heterogeneous solid is (locally) *isotropic*, such that

$$U(\mathbf{x},\boldsymbol{\sigma}) = \psi(\mathbf{x};\tau_e,\sigma_m), \tag{2.6}$$

where ψ is a nonnegative function, that is convex in its last two arguments, and satisfies the condition that $\psi(\mathbf{x};0,0) = 0$ for all $\mathbf{x}$, and where

$$\sigma_m = \frac{1}{3}\mathrm{tr}\,\boldsymbol{\sigma} \quad \text{and} \quad \tau_e = \sqrt{\frac{1}{2}\boldsymbol{\sigma}'\cdot\boldsymbol{\sigma}'}, \tag{2.7}$$

are the *mean* and *effective shear* (in its plasticity denotations) stresses, respectively, with $\boldsymbol{\sigma}' = \boldsymbol{\sigma} - \sigma_m\mathbf{I}$ denoting the stress deviator tensor. We note that form (2.6) is *not* the most general form for the energy function of a nonlinear isotropic solid (we could also have dependence on the determinant of $\boldsymbol{\sigma}$), but this form is still general enough to cover the usual plasticity models of interest here. Further, we will assume that the growth in U as the magnitude of the stress tensor becomes large is stronger that quadratic. This is of course consistent with the ductile behavior of the material.

The new variational principle is obtained then in terms of the following expression for the energy-density function of the heterogeneous solid, namely,

$$U(\mathbf{x},\boldsymbol{\sigma}) = \max_{\mu_o,\kappa_o\geq 0}\left\{U_o(\mathbf{x},\boldsymbol{\sigma}) - V(\mathbf{x};\mu_o,\kappa_o)\right\}, \tag{2.8}$$

where U_o is the energy-density function of a linear-elastic comparison solid with shear modulus μ_o, and bulk modulus κ_o, such that

$$U_o(\mathbf{x},\boldsymbol{\sigma}) = \frac{1}{2\mu_o}\tau_e^2 + \frac{1}{2\kappa_o}\sigma_m^2, \tag{2.9}$$

and where

$$V(\mathbf{x};\mu_o,\kappa_o) = \max_{\boldsymbol{\sigma}}\{U_o(\mathbf{x},\boldsymbol{\sigma}) - U(\mathbf{x},\boldsymbol{\sigma})\}. \tag{2.10}$$

(Note that the maximum in the above function is usually bounded, because of the stronger than quadratic assumption on U). These expressions are obtained by means of the changes of variables, $\nu_e = \tau_e^2$ and $\nu_m = \tau_m^2$, which lead to the definition of a nonnegative function f, such that

$$f(\mathbf{x};\nu_e,\nu_m) = \psi(\mathbf{x};\tau_e,\sigma_m). \tag{2.11}$$

Then, expression (2.10) is nothing more than the Legendre dual of f; in fact, $V(\mathbf{x};\mu_o,\kappa_o) = f^*(\mathbf{x};p_e,p_m)$, with $p_e = \frac{1}{2\mu_o}$ and $p_m = \frac{1}{2\kappa_o}$. Here, f^* is the Legendre transform of f, given by

$$f^*(\mathbf{x};p_e,p_m) = \max_{\nu_e,\nu_m \geq 0}\{p_e\nu_e + p_m\nu_m - f(\mathbf{x};\nu_e,\nu_m)\}, \tag{2.12}$$

and (2.8) is a statement of Legendre duality *for convex* f (*i.e.*, $f^{**} = f$, but written in the terms of U and V). For details, we refer the reader to Ponte Castañeda (1991c).

The new variational principle is then obtained essentially by inserting expression (2.8) for U into the principle of minimum complementary energy (2.4), and interchanging the order of the minimum over the set of admissible stresses with the maximum over the comparison moduli. The result may be expressed in the form (Ponte Castañeda 1991c)

$$\tilde{U}(\overline{\boldsymbol{\sigma}}) = \max_{\mu_o(\mathbf{x}),\kappa_o(\mathbf{x})\geq 0}\left\{\tilde{U}_o(\overline{\boldsymbol{\sigma}}) - \int_\Omega V(\mathbf{x};\mu_o(\mathbf{x}),\kappa_o(\mathbf{x}))\,dx\right\}, \tag{2.13}$$

where

$$\tilde{U}_o(\overline{\boldsymbol{\sigma}}) = \min_{\boldsymbol{\sigma} \in S(\overline{\boldsymbol{\sigma}})} \int_{\Omega} U_o(\mathbf{x}, \boldsymbol{\sigma})\, dx \tag{2.14}$$

is the effective energy of the linear comparison composite. We emphasize that expression (2.13) is a variational principle in its own right since it involves an infinite-dimensional optimization over the set of nonnegative functions $\mu_o(\mathbf{x})$ and $\kappa_o(\mathbf{x})$. Thus, even if we had an explicit expression for the effective energy function of the linear comparison composite $\tilde{U}_o$ (not an easy calculation in general for arbitrary $\mu_o(\mathbf{x})$ and $\kappa_o(\mathbf{x})$), the above variational principle would still be difficult to implement. However, we will see that for a laminated composite, the above problem simplifies dramatically. Similarly, Ponte Castañeda (1991a, b, c) have shown that the above variational principle can also be utilized in an approximate fashion to compute bounds for the effective properties of nonlinear composites with more general microstructures. In the same references, dual versions of (2.13) are also given in terms of the minimum potential energy of the composite; however, in this paper we prefer to use the above formulation due to the fact that it is easier to express the stress/strain relation for a ductile material in terms of the complementary energy-density function U than in terms of its Legendre counterpart, the energy-density function $W = U^*$.

3. Application of the new variational principle to a nonlinear laminated composite

In this section, we specialize the general formulation of the previous section to the case of a laminated composite. Such materials consist of n homogeneous, isotropic phases occupying non-intersecting layered regions $\Omega^{(r)}$ (r = 1, 2,..., n), with union Ω and with normal $\mathbf{n}$. The complementary energy-density function for the laminated material is then expressible in the form

$$U(\boldsymbol{\sigma}, \mathbf{x}) = \sum_{r=1}^{n} \chi^{(r)}(\mathbf{x} \cdot \mathbf{n})\, U^{(r)}(\boldsymbol{\sigma}), \tag{3.}$$

where $\chi^{(r)}(\mathbf{x}\cdot\mathbf{n})$ (equal to 1 for $\mathbf{x}$ in phase r, and 0 otherwise) is the characteristic function of phase r, and $U^{(r)}(\boldsymbol{\sigma}) = \psi^{(r)}(\tau_e, \sigma_m)$ is the corresponding homogeneous, isotropic energy-density function. Also the volume fraction $c^{(r)}$ of each phase is determined by the corresponding characteristic functions $\chi^{(r)}$ via the relation

$$c^{(r)} = \int_{\Omega} \chi^{(r)}(\mathbf{x}\cdot\mathbf{n})\, dx. \tag{3.2}$$

We remark that a laminated composite with *perfectly bonded,* isotropic phases possesses *transversely isotropic* symmetry (with transverse direction $\mathbf{n}$). In some sense, it represents the simplest composite material with transverse isotropy; other examples of practical importance include fiber-reinforced materials with isotropic constituent phases. These will be considered elsewhere. Because of the particular type of anisotropy involved in laminated composites, we have included in Appendix I a brief summary, largely after Walpole (1981), of the appropriate invariants and other useful definitions for transversely isotropic materials.

The computation of the effective energy-density function of a laminated composite is made easy by the following property of laminated composites. If the thickness of the typical layer is small compared to the size of the laminate (*i.e.*, if the laminate is a *composite* in the sense defined in § 2), then, away from a boundary-layer region close to the boundary of the composite, the fields are constant within each layer (a different constant within each layer). Therefore, the problem of determining the effective energy function of a laminated composite reduces to that of determining the constant fields within each phase of the composite by imposition of the appropriate jump conditions (continuity of traction stresses and tangential strains) across the interfaces between the different layers, as well as the averaging conditions stated in § 2. Thus, the problem of determining the effective energy function of a laminated composite, unlike the corresponding problem for a general composite, simplifies to an algebraic one. Although, in principle, the problem can always be solved; in practice, it may be difficult to obtain explicit results because the jump conditions take the form of complicated sets of nonlinear algebraic equations. However, if the composite is made up of linear

phases (with quadratic energy functions in each phase), the jump conditions are also linear and they can be solved in closed form, as discused in § 1.

The results for the effective energy functions of linear laminated composites are given in the next two sections; in the balance of this section, we make use of the new variational principle (2.13) to determine an *exact* expression for the effective energy function of a nonlinear laminated composite $\tilde{U}$, written in terms of the effective energy functions $\tilde{U}_o$ of the class of linear-elastic comparison laminates. This is accomplished by noting that the minimizing comparison moduli functions $\mu_o(\mathbf{x})$ and $\kappa_o(\mathbf{x})$ in (2.13) must be constant within each phase. Therefore, it suffices to optimize with respect to the set of *constant* [over each phase r ($r = 1, ..., n$)] comparison moduli, $\mu_o^{(r)}$ and $\kappa_o^{(r)}$. The new variational principle then leads to

$$\tilde{U}(\overline{\boldsymbol{\sigma}}) = \max_{\mu_o^{(r)}, \kappa_o^{(r)} > 0} \left\{ \tilde{U}_o(\overline{\boldsymbol{\sigma}}) - \sum_{s=1}^{n} c^{(s)} V^{(s)}\left(\mu_o^{(s)}, \kappa_o^{(s)}\right) \right\}, \tag{3.3}$$

where, from (2.10),

$$V^{(r)}\left(\mu_o^{(r)}, \kappa_o^{(r)}\right) = \max_{\tau_e, \sigma_m} \left\{ \frac{1}{2\mu_o^{(r)}} \tau_e^2 + \frac{1}{2\kappa_o^{(r)}} \sigma_m^2 - \psi^{(r)}(\tau_e, \sigma_m) \right\}, \tag{3.4}$$

and where $\tilde{U}_o$ is the effective energy-density function of a linear-elastic laminated material made up of n phases in volume fractions $c^{(r)}$, with shear and bulk moduli, $\mu_o^{(r)}$ and $\kappa_o^{(r)}$, respectively.

On the face of it, expressions (3.3) with (3.4) for the effective energy function of a nonlinear laminated composite do not appear to offer much of an analytical advantage over the standard procedure of determining the stress fields within each phase (by solving the appropriate nonlinear jump conditions) and putting them directly in the complementary energy principle (2.4). This is due to the large number of optimizations involved in expressions (3.3) and (3.4) (*i.e.*, a total of $4n$ optimizations for an n-phase laminate). However, we shall see in the next two sections that application of the particular form for the effective energy function of a linear-elastic laminate in (3.3) leads to a simpler optimization problem for the effective energy function of the nonlinear laminate. Further, we observe that, from a computational point of view, it is generally easier to minimize

functions than it is to solve nonlinear sets of equations, and therefore, the methods developed in this paper are computationally superior to the standard procedure of solving systems of nonlinear equations. In § 4, we begin by considering the simpler case of a laminated composite with incompressible, isotropic phases, and in § 5, we tackle the more complicated problem of a general laminated composite with compressible, isotropic phases.

4. The incompressible laminated composite

In this section, we deal with the special case of laminated composites with incompressible phases. In this case, the energy-density functions of each phase take the simpler form $U^{(r)}(\boldsymbol{\sigma}) = \psi^{(r)}(\tau_e)$. Then, the relations (3.3), (3.4) expressing the effective energy function $\tilde{U}$ of the nonlinear laminate reduce to

$$\tilde{U}(\overline{\boldsymbol{\sigma}}) = \max_{\mu_o^{(r)}>0}\left\{\tilde{U}_o(\overline{\boldsymbol{\sigma}}) - \sum_{s=1}^{n} c^{(s)} V^{(s)}\left(\mu_o^{(s)}\right)\right\}, \tag{4.1}$$

where

$$V^{(r)}\left(\mu_o^{(r)}\right) = \max_{\tau_e}\left\{\frac{1}{2\mu_o^{(r)}}\tau_e^2 - \psi^{(r)}(\tau_e)\right\}, \tag{4.2}$$

and where $\tilde{U}_o$ now refers to the effective energy-density function of a linear-elastic laminated material composed of n incompressible phases in volume fractions $c^{(r)}$, with shear moduli $\mu_o^{(r)}$.

The effective energy-density function of the linear (incompressible) comparison laminate $\tilde{U}_o$ may be computed from the general results of Walpole (1969), specialized to the case of isotropic, incompressible phases. We obtain

$$\tilde{U}_o(\overline{\boldsymbol{\sigma}}) = \frac{1}{2\overline{\mu}_o}\left(\overline{\tau}_p^2 + \overline{\tau}_d^2\right) + \overline{\left(\frac{1}{2\mu_o}\right)}\,\overline{\tau}_n^2, \tag{4.3}$$

where the overbars on the moduli denote volume averages (*e.g.*, $\overline{\mu}_o = \sum_{s=1}^{n} c^{(s)}\mu_o^{(s)}$), and where $\overline{\tau}_p$, $\overline{\tau}_d$, and $\overline{\tau}_n$ are the three transversely isotropic invariants of the applied stress tensor $\overline{\boldsymbol{\sigma}}$ corresponding to the three independent modes for an incompressible, transversely isotropic material (see Appendix I and Figure 1). They are the transverse shear stress, the deviatoric stress, and the longitudinal shear stress, respectively. We note that the three independent modes for a general incompressible, transversely isotropic material reduce to two independent modes for an incompressible laminated composite (since the transverse and deviatoric modes have the same effective response). Note further that, because of the identity $\overline{\tau}_e^2 = \overline{\tau}_p^2 + \overline{\tau}_d^2 + \overline{\tau}_n^2$ from the section on incompressible materials in Appendix I, we are able to rewrite the first term in brackets in (4.3) in the form $\left(\overline{\tau}_e^2 - \overline{\tau}_n^2\right)$.

With expression (4.3) for $\tilde{U}_o$, we can now return to the computation of $\tilde{U}$, implied by (4.1). In this connection, we find that the following identity, proved in Appendix II, is useful in reducing the number of optimizations, namely,

$$\frac{1}{\overline{\mu}_o} = \min_{\omega^{(r)},\,\overline{\omega}=0}\left\{\sum_{s=1}^{n}\frac{c^{(s)}}{\mu_o^{(s)}}\left(1-\omega^{(s)}\right)^2\right\}, \tag{4.4}$$

where the (constant) optimization variables $\omega^{(r)}$ ($r = 1, \ldots, n$) are required to satisfy the constraint $\overline{\omega} = 0$. Then, substituting (4.3), together with (4.4), into (4.1) leads to the result

$$\tilde{U}(\overline{\boldsymbol{\sigma}}) = \max_{\mu_o^{(r)}>0}\ \min_{\omega^{(r)},\,\overline{\omega}=0}\left\{\sum_{s=1}^{n} c^{(s)}\left[\frac{1}{2\mu_o^{(s)}}\left(\tau^{(s)}\right)^2 - V^{(s)}\left(\mu_o^{(s)}\right)\right]\right\}, \tag{4.5}$$

where

$$\tau^{(s)} = \sqrt{\left(\overline{\tau}_p^2 + \overline{\tau}_d^2\right)\left(1-\omega^{(s)}\right)^2 + \overline{\tau}_n^2}\,. \tag{4.6}$$

We note that, by definition, the functions $-V^{(r)}\left(\mu_o^{(r)}\right) = -\left(f^{(r)}\right)^*\left(\frac{1}{2\mu_o^{(r)}}\right)$ are concave in $\frac{1}{\mu_o^{(r)}}$, and similarly the variables $\tau^{(r)}$ are convex in $\omega^{(r)}$. Therefore, by the Saddle Point Theorem (Rockafellar

1970; Corollary 37.3.1), we are allowed to interchange the order of the maximum and the minimum in (4.5). Further, it follows from (2.8) (assuming convexity of $f^{(r)}$, where $f^{(r)}(v_e) = \psi^{(r)}(\tau_e)$ with $v_e = \tau_e^2$) that

$$\psi^{(s)}(\tau) = \max_{\mu_o^{(s)} > 0} \left\{ \frac{1}{2\mu_o^{(s)}} (\tau)^2 - V^{(s)}\left(\mu_o^{(s)}\right) \right\}. \tag{4.7}$$

Therefore, we conclude from (4.5) that

$$\tilde{U}(\overline{\boldsymbol{\sigma}}) = \min_{\omega^{(r)},\, \overline{\omega}=0} \left\{ \sum_{s=1}^{n} c^{(s)} \psi^{(s)}\left(\tau^{(s)}\right) \right\}, \tag{4.8}$$

where the variables $\tau^{(s)}$ are given by relations (4.6). Evidently, this form is much simpler than the original form given by (4.1) and (4.2); it involves an n-dimensional constrained optimization in place of the $2n$-dimensional optimization problem implied by the original form. However, the linear constraint $\overline{\omega} = 0$ for the n optimization variables $\omega^{(r)}$ (r = 1,..., n) can be embedded into the optimization problem (4.8) by letting the n-th variable $\omega^{(n)}$ be expressed in terms of the other $n - 1$ variables $\omega^{(s)}$ (s = 1,..., $n - 1$) via

$$\omega^{(n)} = -\frac{1}{c^{(n)}} \sum_{s=1}^{n-1} c^{(s)} \omega^{(s)}. \tag{4.9}$$

With this modification, the problem (4.8) reduces to an $(n - 1)$-dimensional optimization problem over the unconstrained variables $\omega^{(s)}$ (s = 1,..., $n - 1$). For instance, for the case of a two-phase laminated composite, the problem (4.8) reduces to the one-dimensional optimization problem

$$\tilde{U}(\overline{\boldsymbol{\sigma}}) = \min_{\omega} \left\{ c^{(1)} \psi^{(1)} \left[\sqrt{\left(1 - c^{(2)}\omega\right)^2 \left(\overline{\tau}_e^2 - \overline{\tau}_n^2\right) + \overline{\tau}_n^2} \right] + c^{(2)} \psi^{(2)} \left[\sqrt{\left(1 + c^{(1)}\omega\right)^2 \left(\overline{\tau}_e^2 - \overline{\tau}_n^2\right) + \overline{\tau}_n^2} \right] \right\}, \tag{4.10}$$

which is expressed in terms of one (unconstrained) optimization variable ω. Here, we have made the following identifications, $\omega^{(1)} = c^{(2)}\omega$ and $\omega^{(2)} = -c^{(1)}\omega$.

Finally, we remark that simple expressions for the effective stress/strain relations of the nonlinear transversely isotropic laminated composite may be obtained by means of the results of Appendix III. These relations may be written in terms of the incompressible, transversely isotropic

invariants of the average strain tensor $\bar{\varepsilon}$, namely, the transverse shear strain $\bar{\gamma}_p$, the deviatoric shear strain $\bar{\gamma}_d$, and the longitudinal shear strain $\bar{\gamma}_n$. These strain invariants are defined in Appendix I, and are completely analogous to the corresponding (incompressible) transversely isotropic invariants of the average stress. Thus, with the help of relations (III.7), we may write

$$\bar{\gamma}_p = \left[\sum_{r=1}^{n} c^{(r)}\left(1-\hat{\omega}^{(r)}\right)^2 \frac{1}{\hat{\tau}^{(r)}} \frac{d\psi^{(r)}}{d\tau^{(r)}}\left(\hat{\tau}^{(r)}\right)\right] \frac{\bar{\tau}_p}{2}, \tag{4.11}$$

$$\bar{\gamma}_d = \left[\sum_{r=1}^{n} c^{(r)}\left(1-\hat{\omega}^{(r)}\right)^2 \frac{1}{\hat{\tau}^{(r)}} \frac{d\psi^{(r)}}{d\tau^{(r)}}\left(\hat{\tau}^{(r)}\right)\right] \frac{\bar{\tau}_d}{2},$$

$$\bar{\gamma}_n = \left[\sum_{r=1}^{n} c^{(r)} \frac{1}{\hat{\tau}^{(r)}} \frac{d\psi^{(r)}}{d\tau^{(r)}}\left(\hat{\tau}^{(r)}\right)\right] \frac{\bar{\tau}_n}{2},$$

where $\hat{\tau}^{(r)} = \tau^{(r)}\left(\hat{\omega}^{(r)}\right)$, and where the variables $\hat{\omega}^{(r)}$ are the optimized values of the $\omega^{(r)}$ from (4.8). We note that for the nonlinear laminated composite, there is full coupling between all the distortional (shear) modes. This is different from the situation for the corresponding linear laminated composite [see (4.3)], where all three modes are uncoupled. As we will see in the ensuing discussions, this inter-mode coupling is one of the intrinsic features of laminated (and other anisotropic) nonlinear composites.

5. The compressible laminated composite

With the insight gained in the previous section, we attempt in the present section to obtain corresponding results for n-phase laminated composites with nonlinear, isotropic, compressible phases. In this case, we can apply the results (3.3) and (3.4) of § 3 directly; we only require an expression for the effective energy function $\tilde{U}_o$ of the linear-elastic laminate with isotropic, compressible phases in prescribed volume fractions. This energy function may be computed directly

from the results of Walpole (1969) for the transversely isotropic moduli of linear-elastic laminated composites. The final result may be written in the form

$$\tilde{U}_0(\overline{\boldsymbol{\sigma}}) = \tilde{U}_1(\overline{\boldsymbol{\sigma}}) + \tilde{U}_2(\overline{\boldsymbol{\sigma}}), \tag{5.1}$$

where

$$\tilde{U}_1(\overline{\boldsymbol{\sigma}}) = \frac{1}{2\overline{\mu}_o}\overline{\tau}_p^2 + \overline{\left(\frac{1}{2\mu_o}\right)}\overline{\tau}_n^2, \tag{5.2}$$

$$\tilde{U}_2(\overline{\boldsymbol{\sigma}}) = \frac{1}{2\overline{\eta}_o}\overline{\sigma}_p^2 - \frac{1}{\overline{\eta}_o}\overline{\left(\frac{3\kappa_o - 2\mu_o}{3\kappa_o + 4\mu_o}\right)}\overline{\sigma}_p\,\overline{\sigma}_n + \frac{1}{2}\left[\overline{\left(\frac{3}{3\kappa_o + 4\mu_o}\right)} + \frac{1}{\overline{\eta}_o}\overline{\left(\frac{3\kappa_o - 2\mu_o}{3\kappa_o + 4\mu_o}\right)}^2\right]\overline{\sigma}_n^2, \tag{5.3}$$

with $\eta_o = 9\kappa_o\mu_o/(3\kappa_o + 4\mu_o)$, and where $\overline{\sigma}_p$, $\overline{\sigma}_n$, $\overline{\tau}_p$ and $\overline{\tau}_n$ are the four transversely isotropic invariants (up to quadratic in order) of the applied stress $\overline{\boldsymbol{\sigma}}$. They denote, respectively, the in-plane hydrostatic stress, the normal tensile stress, the transverse shear stress and the longitudinal shear stress (see Appendix I and Figure 1). The reason behind the above splitting of $\tilde{U}_o$ lies in the similarity between the first part of (5.1), as given by (5.2), for the distortional (shear) modes of the compressible laminate and relation (4.3) for the incompressible composite (with $\overline{\tau}_p^2 + \overline{\tau}_d^2$ replaced by $\overline{\tau}_p^2$). Thus, it follows immediately that

$$\tilde{U}_1(\overline{\boldsymbol{\sigma}}) = \min_{\omega_e^{(r)},\,\overline{\omega}_e = 0}\left\{\sum_{s=1}^{n}\frac{c^{(s)}}{2\mu_o^{(s)}}\left[\left(1-\omega_e^{(s)}\right)^2\overline{\tau}_p^2 + \overline{\tau}_n^2\right]\right\}, \tag{5.4}$$

where the $\omega_e^{(r)}$ are the corresponding optimization variables, and they are subject to the constraint $\overline{\omega}_e = 0$. The second part is more complicated, but it can be shown by straightforward computation that, if $\overline{\sigma}_p \neq 0$, $\tilde{U}_2(\overline{\boldsymbol{\sigma}})$ may be represented in the form

$$\tilde{U}_2(\overline{\boldsymbol{\sigma}}) = \min_{\omega_m^{(r)},\,\overline{\omega}_m = 0}\left\{\sum_{s=1}^{n}\frac{c^{(s)}}{6\mu_o^{(s)}}\left[\overline{\sigma}_n - \left(1-\omega_m^{(s)}\right)\overline{\sigma}_p\right]^2 + \sum_{s=1}^{n}\frac{c^{(s)}}{2\kappa_o^{(s)}}\left[\tfrac{1}{3}\overline{\sigma}_n + \tfrac{2}{3}\left(1-\omega_m^{(s)}\right)\overline{\sigma}_p\right]^2\right\}, \tag{5.5}$$

where the optimization variables $\omega_m^{(r)}$ are also subject of the constraint $\overline{\omega}_m = 0$.

By putting together relations (5.4) and (5.5), we arrive at the following expression for the linear comparison laminate

$$\tilde{U}_o(\bar{\boldsymbol{\sigma}}) = \min_{\substack{\omega_e^{(r)},\,\bar{\omega}_e=0 \\ \omega_m^{(r)},\,\bar{\omega}_m=0}} \left\{ \sum_{s=1}^{n} \frac{c^{(s)}}{2\mu_o^{(s)}} \left(\tau_e^{(s)}\right)^2 + s \sum_{r=1}^{n} \frac{c^{(s)}}{2\kappa_o^{(s)}} \left(\sigma_m^{(s)}\right)^2 \right\}, \tag{5.6}$$

where

$$\tau_e^{(s)} = \sqrt{\left(1-\omega_e^{(s)}\right)^2 \bar{\tau}_p^2 + \bar{\tau}_n^2 + \tfrac{1}{3}\left[\bar{\sigma}_n - \left(1-\omega_m^{(s)}\right)\bar{\sigma}_p\right]^2}\,, \text{ and } \quad \sigma_m^{(s)} = \tfrac{1}{3}\bar{\sigma}_n + \tfrac{2}{3}\left(1-\omega_m^{(s)}\right)\bar{\sigma}_p. \tag{5.7}$$

We note that this result is reminiscent of the type of result that one would expect to arise directly from the principle of minimum complementary energy. That this result is indeed directly obtainable from the principle of minimum complementary energy is demonstrated in Appendix IV.

Then, following a procedure similar to the one followed in the development of expression (4.8) for the effective energy function of the nonlinear, incompressible laminated composite, but making use of (3.3) and (3.4), we arrive at the following expression for the effective energy function of the nonlinear laminated composite

$$\tilde{U}(\bar{\boldsymbol{\sigma}}) = \min_{\substack{\omega_e^{(r)},\,\omega_m^{(r)} \\ \bar{\omega}_e=\bar{\omega}_m=0}} \left\{ \sum_{s=1}^{n} c^{(s)} \psi^{(s)}\left(\tau_e^{(s)}, \sigma_m^{(s)}\right) \right\}, \tag{5.8}$$

where $\tau_e^{(s)}$ and $\sigma_m^{(s)}$ are given by (5.7). Here, we have made use of the Saddle Point theorem allowing the interchange in the order of the minimum over the $\omega_e^{(r)}$, $\omega_m^{(r)}$ variables with the maximum over the comparison moduli $\mu_o^{(r)}$ and $\kappa_o^{(r)}$. We note that form (5.8) for the effective energy function $\tilde{U}$ of the nonlinear laminated composite is a direct generalization of form (5.6) for the effective energy function $\tilde{U}_o$ of the linear comparison laminate. There are, of course, sound physical reasons for this similarity, and these are discussed in Appendix IV, where it is shown that the procedure used in the derivation of (5.6) for the linear laminate can be extended to give an alternative derivation of (5.8) for the nonlinear laminate. In Appendix IV, we also show that an alternative form of (5.8) is possible, which is not subject to the $\bar{\sigma}_p \neq 0$ restriction, although we

should emphasize that the above form is valid in the limit as $\overline{\sigma}_p \to 0$ (it is just not valid in a pointwise sense at $\overline{\sigma}_p = 0$, because the optimizing variables $\omega_m^{(r)}$ become unbounded in that limit).

We note that while the distortional and dilatational modes are not coupled in the linear laminated composite [although the dilational modes are coupled among themselves, as (5.3) shows], the same is not true of the nonlinear laminated composite, as relations (5.8) and $(5.7)_1$ clearly demonstrate.

The new representation for the effective energy function of a nonlinear laminated composite $\tilde{U}$ can be seen to involve only a $2n$-dimensional optimization problem with two linear constraints. This is major reduction in order compared with the original expressions (3.3) and (3.4) involving a $4n$-dimensional optimization problem. However, as noted in the previous section, further reductions are possible [to a $2(n-1)$-dimensional optimization problem] by embedding the constraints directly into the optimization problem (5.8). For example, for the case of a two phase composite, we obtain a result involving only a two-dimensional optimization problem prescribed in terms of the variables ω_e, ω_m via

$$\tilde{U}(\overline{\boldsymbol{\sigma}}) = \min_{\omega_e,\omega_m} \left\{ c^{(1)} \psi^{(1)}\left(\tau_e^{(1)}, \sigma_m^{(1)}\right) + c^{(2)} \psi^{(2)}\left(\tau_e^{(2)}, \sigma_m^{(2)}\right) \right\}, \tag{5.9}$$

where $\tau_e^{(1)}, \sigma_m^{(1)}$ and $\tau_e^{(2)}, \sigma_m^{(2)}$ are given by relations (5.7) with $\omega_e^{(1)} = c^{(2)}\omega_e$, $\omega_e^{(2)} = -c^{(1)}\omega_e$, $\omega_m^{(1)} = c^{(2)}\omega_m$, and $\omega_m^{(2)} = -c^{(1)}\omega_m$.

Finally, we remark that simple expressions for the effective stress/strain relations of the transversely isotropic laminated composite may be obtained by means of the results of Appendix III. These may be written in terms of the transversely isotropic invariants of the average strain tensor $\overline{\boldsymbol{\varepsilon}}$, the in-plane hydrostatic strain $\overline{\varepsilon}_p$, the normal tensile strain $\overline{\varepsilon}_n$, the transverse shear strain $\overline{\gamma}_p$, and the longitudinal shear strain $\overline{\gamma}_n$. These are defined in Appendix I, and are completely analogous to the corresponding transversely isotropic invariants of the average stress. Thus, with the help of relations (III.8), we may write

$$(5.10)\qquad \bar{\varepsilon}_p = \frac{1}{6}\sum_{r=1}^{n} c^{(r)}\left(1-\hat{\omega}_m^{(r)}\right)\left[2\frac{\partial \psi^{(r)}}{\partial \sigma_m^{(r)}}\left(\hat{\tau}_e^{(r)},\hat{\sigma}_m^{(r)}\right)+\left[\left(1-\hat{\omega}_m^{(r)}\right)\bar{\sigma}_p-\bar{\sigma}_n\right]\frac{1}{\hat{\tau}_e^{(r)}}\frac{\partial \psi^{(r)}}{\partial \tau_e^{(r)}}\left(\hat{\tau}_e^{(r)},\hat{\sigma}_m^{(r)}\right)\right],$$

$$\bar{\varepsilon}_n = \frac{1}{3}\sum_{r=1}^{n} c^{(r)}\left[\frac{\partial \psi^{(r)}}{\partial \sigma_m^{(r)}}\left(\hat{\tau}_e^{(r)},\hat{\sigma}_m^{(r)}\right)-\left[\left(1-\hat{\omega}_m^{(r)}\right)\bar{\sigma}_p-\bar{\sigma}_n\right]\frac{1}{\hat{\tau}_e^{(r)}}\frac{\partial \psi^{(r)}}{\partial \tau_e^{(r)}}\left(\hat{\tau}_e^{(r)},\hat{\sigma}_m^{(r)}\right)\right],$$

$$\bar{\gamma}_p = \left[\sum_{r=1}^{n} c^{(r)}\left(1-\hat{\omega}_e^{(r)}\right)^2\frac{1}{\hat{\tau}_e^{(r)}}\frac{\partial \psi^{(r)}}{\partial \tau_e^{(r)}}\left(\hat{\tau}_e^{(r)},\hat{\sigma}_m^{(r)}\right)\right]\frac{\bar{\tau}_p}{2},$$

$$\bar{\gamma}_n = \left[\sum_{r=1}^{n} c^{(r)}\frac{1}{\hat{\tau}_e^{(r)}}\frac{\partial \psi^{(r)}}{\partial \tau_e^{(r)}}\left(\hat{\tau}_e^{(r)},\hat{\sigma}_m^{(r)}\right)\right]\frac{\bar{\tau}_n}{2},$$

where $\hat{\tau}_e^{(r)} = \tau_e^{(r)}\left(\hat{\omega}_e^{(r)},\hat{\omega}_m^{(r)}\right)$, $\hat{\sigma}_m^{(r)} = \sigma_m^{(r)}\left(\hat{\omega}_m^{(r)}\right)$, and where $\hat{\omega}_e^{(r)}$, $\hat{\omega}_m^{(r)}$ are the optimized values of $\omega_e^{(r)}$, $\omega_m^{(r)}$ from (5.8). We note that in this form, the coupling between the distortional and dilatational modes in the nonlinear material become evident since $\bar{\gamma}_p$, $\bar{\gamma}_n$ depend on $\bar{\sigma}_n$, $\bar{\sigma}_p$, and, conversely, $\bar{\varepsilon}_n$, $\bar{\varepsilon}_p$ depend on $\bar{\tau}_n$, $\bar{\tau}_p$.

6. Application to laminated composites with power-law constitutive behavior

In this section, we specialize the results of § 4 and § 5 for three classes of laminated composites. The first subsection deals with the case of an incompressible laminated material made up of layers of a phase with "linear plus power-hardening" constitutive behavior, reinforced with stiffer layers of a linear-elastic material. In the study of these incompressible laminates, we will emphasize the coupling between different distortional loading modes arising as a consequence of nonlinearity and anisotropy in the laminates. The second subsection is dedicated to the study of a compressible, aluminum/alumina laminate, and the understanding of the dilatational modes is emphasized in this case. The third subsection deals with an incompressible laminated composite made up of two

rigid/perfectly plastic phases with different yield stresses; it is interesting to note that, in this special case, completely explicit results are obtained for the effective yield function of the laminate.

6.1. *Incompressible laminated composites*

In this subsection, we consider an incompressible, two-phase laminated composite characterized by the following constitutive laws for the two isotropic phases. Phase #1 is governed by "linear plus power-hardening" constitutive behavior described by the energy-density function

$$\psi^{(1)}(\tau_e) = \int_o^{\sqrt{3}\tau_e} F^{(1)}(s)\,ds, \tag{6.1}$$

where

$$F^{(1)}(s) = \varepsilon_o \left\{ \frac{s}{\sigma_o} + \left[\left(\frac{s}{\sigma_o} \right)^n - \left(\frac{\sigma_y}{\sigma_o} \right)^n \right] H(s - \sigma_y) \right\}. \tag{6.2}$$

Here H is the unit step function (equal to 0 when $s \le \sigma_y$ and to 1 otherwise), and ε_o, σ_o are strain, stress normalization factors such that $\sigma_o/\varepsilon_o = 3\mu^{(1)}$, with $\mu^{(1)}$ denoting the shear modulus of phase #1. Then, the function $F^{(1)}$ represents the uniaxial stress/strain relation of phase #1 under simple tension loading conditions. Thus, the behavior of phase #1 is linear when the uniaxial stress is lower than the yield stress, σ_y, and is linear plus power-hardening for stresses larger than σ_y. The factor $\sqrt{3}$ in (6.1) is needed in order to fit the isotropic stress invariant τ_e to the uniaxial case. Phase #2 is linear and governed by the quadratic energy-density function

$$\psi^{(2)}(\tau_e) = \frac{1}{2\mu^{(2)}} \tau_e^2, \tag{6.3}$$

where $\mu^{(2)}$ is the shear modulus of the phase.

With the above constitutive behavior for the two phases (# 1 and 2), which are prescribed in volume fractions $(1 - c^{(2)})$ and $c^{(2)}$, respectively, the effective energy-density function of the incompressible laminated composite may be expressed in dimensionless form via the relation

$$\frac{\tilde{U}(\bar{\boldsymbol{\sigma}})}{\tau_o \gamma_o} = G\left\{ \frac{\bar{\tau}_p}{\tau_o}, \frac{\bar{\tau}_d}{\tau_o}, \frac{\bar{\tau}_n}{\tau_o}; \frac{\sigma_y}{\sigma_o}, \frac{\mu^{(2)}}{\mu^{(1)}}, n, c^{(2)} \right\}, \tag{6.4}$$

where the specific form of the function G is determined from (4.10), and $\tau_o = \sqrt{\frac{1}{3}}\sigma_o$, $\gamma_o = \sqrt{\frac{3}{4}}\varepsilon_o$, so that $\tau_o/\gamma_o = 2\mu^{(1)}$. Then, the relations between the three (incompressible) transversely isotropic stress invariants and the corresponding strain invariants may be computed from (4.11). These relations are presented in Figures 2 through 5 for the following values of the four parameters appearing in (6.4):

$$\frac{\sigma_y}{\sigma_o} = 1, \quad \frac{\mu^{(2)}}{\mu^{(1)}} = 5, \quad n = 3 \text{ and } c^{(2)} = 0.2.$$

We recall that there are only two independent modes for the incompressible laminated composites; they are the longitudinal shear stress $\bar{\tau}_n$ and the following combination of the other two shear modes $\sqrt{\bar{\tau}_p^2 + \bar{\tau}_d^2}$ (*i.e.* the transverse and deviatoric shear modes, respectively). For simplicity, we will refer to this combination of the two modes as $\bar{\tau}$ and to the corresponding combination of the strain modes, $\sqrt{\bar{\gamma}_p^2 + \bar{\gamma}_d^2}$, as $\bar{\gamma}$. Thus, it suffices to consider the relations among the stress modes $\bar{\tau}_n$, $\bar{\tau}$ and the strain modes $\bar{\gamma}_n$, $\bar{\gamma}$ in order to have a complete description of the constitutive behavior of the incompressible laminate. In order to highlight the effect of nonlinearity, results are included in Figures 2 to 5 in the form of short-dashed curves for a linear laminated composite with the same shear moduli as the nonlinear laminate. Thus, the phases of this linear *reference* laminate are similar to those of the nonlinear one with the only difference that in phase #1 $\sigma_y = \infty$.

Figure 2 shows a plot of the longitudinal shear stress $\bar{\tau}_n$ versus the longitudinal shear strain $\bar{\gamma}_n$ for three different values of $\bar{\tau}$ ($\bar{\tau}/\tau_o = 0, 2, 5$). We observe that when there is no preloading of the laminate ($\bar{\tau}/\tau_o = 0$), the behavior of the stress/strain curve of the nonlinear laminate is initially the same as that of the reference linear laminate (short-dash line) until phase # 1 yields. After yielding, the two curves diverge with the nonlinear phase controlling the behavior for large longitudinal shear stresses. That this should be so is seen from the fact that shear parallel to the layers should be controlled by the less stiff phase (in this case, the nonlinear phase). The effect of increasing $\bar{\tau}$ is to saturate the linear range of phase # 1, making the effective stress/strain curve of

the laminate be controlled by the nonlinear phase even for small values of the longitudinal shear stresses $\bar{\tau}_n$.

Figure 3 shows a plot of $\bar{\tau}$ versus the longitudinal shear strain $\bar{\gamma}_n$ for two different values of the longitudinal shear stress ($\bar{\tau}_n/\tau_o = 0.5, 1$), and serves to emphasize the coupling between the two shear modes. Thus, a small preload in the form of a longitudinal shear stress $\bar{\tau}_n$ applied to the nonlinear laminate can lead to large increases in the longitudinal strain $\bar{\gamma}_n$ as the other shear stress mode $\bar{\tau}$ is increased; in fact, the growth is unbounded and can be shown to be proportional to $(\bar{\tau}/\tau_o)^{\frac{n-1}{n}}$.

Figure 4 shows a plot of the shear stress $\bar{\tau}$ versus the shear strain $\bar{\gamma}$ for three different values of longitudinal shear stress $\bar{\tau}_n$ ($\bar{\tau}_n/\tau_o = 0, 0.5, 1$). We observe that when there is no preloading of the laminate ($\bar{\tau}_n/\tau_o = 0$), the behavior of the effective stress/strain curve of the nonlinear laminate is initially the same as that of the reference linear laminate (short-dash line) until phase # 1 yields. After yielding, however, the two curves diverge with the linear phase controlling the behavior for large shear stresses $\bar{\tau}$. In fact, it can be demonstrated that the slope of the stress/strain curve in question reaches an asymptotic value of $2c^{(2)}\mu^{(2)}$ (corresponding to a linear Voigt estimate with $\mu^{(1)} \to 0$) as the shear stress $\bar{\tau}$ becomes large. Evidently, the weaker nonlinear phase is acting as if it was not present for large enough $\bar{\tau}$. The effect of increasing $\bar{\tau}_n$ is then to saturate the linear range of phase # 1, reducing the effect of the nonlinear phase on the effective stress/strain curve of the composite (the laminate behaves almost linearly with modulus $c^{(2)}\mu^{(2)}$ for sufficiently large preload $\bar{\tau}_n$). That the nonlinear laminate should be controlled by the stiffer linear phase for large magnitudes of the transverse shear stress $\bar{\tau}_p$ (and fixed longitudinal shear stress $\bar{\tau}_n$) is easy to visualize, but that exactly the same behavior should be observed for the deviatoric mode $\bar{\tau}_d$ (the other component of $\bar{\tau}$) is perhaps less intuitive. The reason, however, is related to the Poisson effect. Thus, for example, if the laminate is compressed along the normal direction (which may seem to be controlled by the less stiff nonlinear phase), tensile strains are set up in the plane of the layers,

which must be continuous across the phases, thus providing the required stiffening effect in the normal direction (because the linear phase controls the in-plane behavior of the laminate).

Figure 5 shows the relation between the longitudinal shear stress mode $\bar{\tau}_n$ and the strain mode $\bar{\gamma}$ for different values of the shear stress $\bar{\tau}$ $(\bar{\tau}/\tau_o = 0.5, 2)$. We observe that while there is significant coupling between the two modes (by comparison with the linear reference laminate), the coupling is not as significant as in Figure 3. Thus, the shear strain $\bar{\gamma}$ reaches a maximum level for a given shear preload $\bar{\tau}$ as the longitudinal shear $\bar{\tau}_n$ is increased. This is because the nonlinear phase is dominated by the linear phase in this mode of deformation as observed previously in connection with Figure 4. The effect of increasing preload $\bar{\tau}$ is to increase (in both absolute and relative terms) the increments in the shear strain $\bar{\gamma}$ with increasing shear stress $\bar{\tau}_n$.

6.2. *The aluminum/alumina laminated composite*

In this subsection, we demonstrate the behavior of a nonlinear, compressible, laminated composite made up of aluminum layers reinforced with layers of alumina. Aluminum is a ductile material with uniaxial stress/strain curves that can be approximated by a "linear-plus-power" law with hardening exponent n varying between 4.2 and 5.8. Thus, we will assume the following form for the energy-density function of the aluminum layers (phase # 1)

$$\psi^{(1)}(\tau_e, \sigma_m) = \int_o^{\sqrt{3}\tau_e} F^{(1)}(s)\,ds + \frac{1}{2\kappa^{(1)}}\sigma_m^2, \tag{6.5}$$

where $F^{(1)}$ is the same as in (6.2), and thus the only difference between (6.1) and (6.5) is the compressibility of aluminum accounted for in (6.5) through the bulk modulus $\kappa^{(1)}$. Alumina (phase # 2) is a brittle material that behaves in a linear fashion up to the point of failure. Its energy-density function is represented by

$$\psi^{(2)}(\tau_e, \sigma_m) = \frac{1}{2\mu^{(2)}}\tau_e^2 + \frac{1}{2\kappa^{(2)}}\sigma_m^2, \tag{6.6}$$

where $\mu^{(2)}$ and $\kappa^{(2)}$ denote the shear and bulk moduli of the alumina, respectively.

With this choice of $\psi^{(1)}$ and $\psi^{(2)}$ (for the behaviors of the two phases), the effective energy-density function of the composite can be represented in dimensionless form via

$$\frac{\tilde{U}(\overline{\boldsymbol{\sigma}})}{\tau_o \gamma_o} = G\left\{\frac{\overline{\sigma}_p}{\sigma_o}, \frac{\overline{\sigma}_n}{\sigma_o}, \frac{\overline{\tau}_p}{\tau_o}, \frac{\overline{\tau}_n}{\tau_o}; \frac{\mu^{(2)}}{\mu^{(1)}}, \nu^{(1)}, \nu^{(2)}, n, c^{(2)}\right\}, \tag{6.7}$$

where G is obtained from (5.9), $\tau_o = \sqrt{\frac{1}{3}}\sigma_o$, $\gamma_o = \sqrt{\frac{3}{4}}\varepsilon_o$, such that $\tau_o/\gamma_o = 2\mu^{(1)}$, and $\nu^{(1)}$, $\nu^{(2)}$ are the (dimensionless) Poisson's ratios of the two phases defined by

$$\nu^{(r)} = \frac{3\kappa^{(r)} - 2\mu^{(r)}}{6\kappa^{(r)} + 2\mu^{(r)}} .$$

In the results to follow, we will make the following choices (which are representative of the aluminum/alumina composite) for the material parameters in (6.7):

$$\frac{\mu^{(2)}}{\mu^{(1)}} = 6, \quad \nu^{(1)} = 0.35, \quad \nu^{(2)} = 0.25 \text{ and } n = 5.$$

The results are presented in Figures 6 through 9 in terms of plots of the four transversely isotropic stress modes versus the corresponding strain modes for three different values of the volume fraction of alumina $c^{(2)}$ (0.1, 0.25 and 0.5).

Figure 6 shows a plot of the in-plane hydrostatic stress $\overline{\sigma}_p$ versus the corresponding hydrostatic strain $\overline{\varepsilon}_p$, when all other stress modes vanish, for the three values of the volume fraction of alumina $c^{(2)}$. We observe that the laminate has a linear range with effective modulus $2\overline{\eta}$ (recall that $\eta = 9\kappa\mu/(3\kappa + 4\mu)$) up to yielding of the aluminum phase. However, the laminate behaves almost linearly even after yielding with modulus approaching $2c^{(2)}\eta^{(2)}$ for large values of $\overline{\sigma}_p$. This behavior is expected on physical grounds due to the fact that the stiffer material (alumina) should dominate the behavior in tension (compression) parallel to the layers. The effect of increasing volume fractions of alumina is of course to stiffen the effective behavior of the composite.

Figure 7 shows a plot of the normal tensile stress $\overline{\sigma}_n$ versus the corresponding tensile strain $\overline{\varepsilon}_n$, when all other stress modes vanish, for the three values of the volume fraction of alumina $c^{(2)}$. The structure of the plots is very similar to that of Figure 6; however, the effective moduli are

different. Before phase # 1 reaches yielding, the laminate has uniaxial modulus given by the expression

$$\left[\overline{\left(\frac{3}{3\kappa+4\mu}\right)}+\frac{1}{\overline{\eta}}\overline{\left(\frac{3\kappa-2\mu}{3\kappa+4\mu}\right)^2}\right]^{-1},$$

while after yielding of phase # 1, the modulus for large stress $\overline{\sigma}_n$ is reduced to the level

$$\left[\frac{1}{c^{(2)}\eta^{(2)}}+\overline{\left(\frac{1}{\kappa}\right)}-\frac{4}{3\kappa^{(2)}}\right]^{-1}.$$

In this case, it is not evident that the linear phase should govern the effective behavior of the laminate for large stresses. The reason, however, is the same as discussed in the previous subsection in connection with Figure 4: continuity of the tangential strains across the interfaces together with the Poisson effect.

Figure 8 shows the corresponding plots for the transverse shear stress $\overline{\tau}_p$ versus the transverse shear strain $\overline{\gamma}_p$, when no other stress modes are present, for the three values of the volume fraction of alumina. In this case, the results are similar to those of Figure 6 for clear physical reasons: the stiffer phase controls the behavior of the laminate under transverse shear loading.

Figure 9 shows plots of the longitudinal shear stress $\overline{\tau}_n$ versus the corresponding strain mode $\overline{\gamma}_n$, with no other stress modes present. The behavior in this case is dramatically different, as the study of the corresponding case for the incompressible laminate demonstrated earlier (Figure 2). Thus, after an initial linear range before yielding of phase # 1, the weaker nonlinear phase governs the effective behavior of the laminate. In contrast to the other three modes, we observe that the dependence on the volume fraction of alumina is fairly weak, so that the three curves (corresponding to different values of $c^{(2)}$) are quite close to each other.

Clearly, a study of the inter-relations between the different modes would be required to have a complete picture of the effective behavior of the nonlinear compressible laminate. However, the behavior of these inter-modal relations is similar to those already explored for the incompressible laminate. Thus, the inter-modes relations that involve the longitudinal shear strain $\overline{\gamma}_n$ are of the form

of the relations presented in Figure 3 while all other inter-modal stress/strain relations are in the form of Figure 5.

6.3. *The rigid/perfectly plastic laminated composite*

In this subsection, we consider the case of an incompressible laminated composite made up of two rigid/perfectly plastic phases with yield stresses $\tau_o^{(1)}$ and $\tau_o^{(2)}$, chosen such that $\tau_o^{(1)} < \tau_o^{(2)}$, in given volume fractions $c^{(1)}$ and $c^{(2)}$. The behavior of the phases may then be characterized in terms of the convex energy-density functions

$$\psi^{(r)}(\tau_e) = \begin{cases} 0, & \tau_e \le \tau_o^{(r)}, \\ \infty, & \tau_e > \tau_o^{(r)}, \end{cases} \tag{6.8}$$

(r =1, 2), where τ_e denotes the effective shear stress. These energy functions may be obtained directly from pure power-law energy functions of the form

$$\psi^{(r)}(\tau_e) = \frac{1}{n+1}\varepsilon_o\sigma_o\left(\frac{\tau_e}{\tau_o^{(r)}}\right)^{n+1}, \tag{6.9}$$

in the limit as $n \to \infty$. Further, these energy functions define "yield functions" for the phase materials that may be described in the usual way via

$$\phi^{(r)}(\boldsymbol{\sigma}) = \tau_e - \tau_o^{(r)} = 0. \tag{6.10}$$

Here, we will proceed *formally* and make use of expression (4.10) to determine an expression for the effective energy function of the laminated material $\tilde{U}$, from which we will be able to determine a yield function for the laminated composite Φ. For a rigorous treatment of homogeneization theory for rigid/perfectly plastic composites, and in particular for a discussion concerning the validity of the normality condition for the effective yield function of the composite, the reader is referred to Suquet (1985). Because of incompressibility and transverse isotropy, we will find that such a yield function may be represented as a curve in the $(\bar{\tau}_n, \bar{\tau})$-space, where $\bar{\tau} = \sqrt{\bar{\tau}_p^2 + \bar{\tau}_d^2}$. Thus, application of (4.10) leads to the expression

$$\tilde{U}(\bar{\boldsymbol{\sigma}}) = \min_{\omega}\left\{G(\bar{\tau}_p, \bar{\tau}_n, \omega)\right\}, \tag{6.11}$$

where

$$G = \begin{cases} 0, & \tau^{(1)} \le \tau_o^{(1)} \text{ and } \tau^{(2)} \le \tau_o^{(2)}, \\ \infty, & \tau^{(1)} > \tau_o^{(1)} \text{ or } \tau^{(2)} > \tau_o^{(2)}, \end{cases} \tag{6.12}$$

and where $\tau^{(1)} = \sqrt{\left(1 - c^{(2)}\omega\right)^2 \bar{\tau}^2 + \bar{\tau}_n^2}$ and $\tau^{(2)} = \sqrt{\left(1 + c^{(1)}\omega\right)^2 \bar{\tau}^2 + \bar{\tau}_n^2}$.

The above optimization problem for ω then reduces to determining all possible combinations of $\bar{\tau}$ and $\bar{\tau}_n$ for which $\tilde{U} = 0$, which in turn defines the yield function for the composite Φ. First, we note that, independent of $\bar{\tau}$ and ω, $\tilde{U}$ can only vanish if

$$\bar{\tau}_n - \tau_o^{(1)} \le 0, \tag{6.13}$$

for otherwise $\tau^{(1)} \ge \bar{\tau}_n > \tau_o^{(1)}$. Thus, inequality (6.13) is a *necessary* condition for $\tilde{U}$ to vanish. However, the condition (6.13) is not *sufficient* to ensure that $\tilde{U}$ vanishes since the condition $\tau^{(2)} \le \tau_o^{(2)}$ may be violated. Thus, assuming that condition (6.13) is satisfied, we ask the question of whether there are values of ω, depending on $\bar{\tau}$ and $\bar{\tau}_n$, such that conditions $\tau^{(1)} \le \tau_o^{(1)}$ and $\tau^{(2)} \le \tau_o^{(2)}$ are satisfied simultaneously. The answer is affirmative, provided that $\bar{\tau}$ and $\bar{\tau}_n$ (for given volume fractions $c^{(1)}$ and $c^{(2)}$) satisfy the condition

$$\bar{\tau} \le c^{(1)}\sqrt{\left(\tau_o^{(1)}\right)^2 - \bar{\tau}_n^2} + c^{(2)}\sqrt{\left(\tau_o^{(2)}\right)^2 - \bar{\tau}_n^2}. \tag{6.14}$$

Thus, conditions (6.13) and (6.14) define an effective yield function for the composite, $\Phi = 0$, such that

$$\Phi(\bar{\sigma}) \equiv \begin{cases} \bar{\tau} - \left[c^{(1)}\sqrt{\left(\tau_o^{(1)}\right)^2 - \bar{\tau}_n^2} + c^{(2)}\sqrt{\left(\tau_o^{(2)}\right)^2 - \bar{\tau}_n^2}\right], & \bar{\tau}_n < \tau_o^{(1)}, \\ \bar{\tau}_n - \tau_o^{(1)}, & \bar{\tau}_n = \tau_o^{(1)}. \end{cases} \tag{6.15}$$

We note that when $\tau_o^{(2)} = \tau_o^{(1)}$, the expression above reduces to the von Mises yield criterion.

Plots of the yield surfaces in the $(\bar{\tau}_n, \bar{\tau})$-space of applied stresses are given in Figures 10 and 11. Figure 10 shows the exact yield surface Φ for the choice of parameters, $\tau_o^{(2)}/\tau_o^{(1)} = 2$ and $c^{(2)} = 0.5$. The isotropic Reuss and Voigt (also known as Bishop-Hill estimate) bounds for the yield surfaces are also given for comparison. We note that the exact yield surface Φ is close to the Voigt upper bounding surface Φ_V for low values of the longitudinal shear stress ($\bar{\tau}_n < \frac{1}{2}\tau_o^{(1)}$), and close to

the Reuss lower bounding surface Φ_R for low values of $\overline{\tau}$. We also include in Figure 10 an estimate for the yield surface, Φ_H, which is based on the approximation of Hill (1951) for slightly anisotropic materials. This approximate yield surface is given by

$$\Phi_H(\overline{\sigma}) \equiv \frac{\overline{\tau}^2}{\left(c^{(1)}\tau_o^{(1)} + c^{(2)}\tau_o^{(2)}\right)^2} + \frac{\overline{\tau}_n^2}{\left(\tau_o^{(1)}\right)^2} - 1, \tag{6.16}$$

and we note that it amounts to an elliptic interpolation between the Voigt and Reuss yield functions.

We observe that for the largely anisotropic case depicted in Figure 10 ($\tau_o^{(2)}/\tau_o^{(1)} = 2$ and $c^{(2)} = 0.5$), Hill's elliptic approximation severely underestimates the ultimate yield strenght of the laminated composite for combined longitudinal and transverse loading. Figure 11 shows plots of the exact yield surfaces (continuous lines) and Hill's approximate yield surfaces (dashed lines) for a laminated composite with slight anisotropy ($\tau_o^{(2)}/\tau_o^{(1)} = 1.25$) and three values of $c^{(2)}$ (0.1, 0.5 and 0.9). For all values of $c^{(2)}$, the exact yield criterion bounds a larger region of the $(\overline{\tau}_n, \overline{\tau})$-plane than the Hill approximate criterion, and the two curves are only in good agreement for small volume fractions of the stronger phase ($c^{(2)} = 0.1$).

6. Closure

In this paper, we have described the application of a new variational method, developed by Ponte Castañeda (1991a, c), to a laminated composite material with elastoplastic phases in prescribed volume fractions. It constitutes one of the first applications of the method to composite materials with anisotropic symmetries (see also Ponte Castañeda 1991c for corresponding results for fiber-reinforced materials). Because of the simplicity of the laminated microstructure, allowing for the determination of the *exact* effective properties of the laminated composite, this work is of interest not only on account of the practical significance of laminated composites themselves, but also because it provides a case study where the power of the method can be evaluated. Additionally, the results of this paper suggest that when dealing with strongly anisotropic materials, it is not sufficient to consider the behavior of the composite under special loading conditions, since the behavior of the

composite may be *dramatically* different under a different type of loading conditions. Thus, we found that nonlinearity highlighted the differences between the constitutive response of the laminated composite under transverse and longitudinal shear loading. Further, the study of the nonlinear laminated composites also underlined the significant coupling that may arise between different loading modes (which does not take place for linear composites with similar microstructures). Thus, it was found that a small fixed preload of the laminate in the longitudinal direction leads to continued increase of the longitudinal shear strain as the transverse shear stress (for example) is incremented. It is anticipated that the features uncovered by the present analysis of nonlinear laminated composites will also be important in other types of nonlinear composites with anisotropic symmetries, such as the practically important class of fiber-reinforced materials.

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Appendix I: On the characterization of transversely isotropic materials

The purpose of this appendix is to gather some results relevant to the analysis of linear-elastic materials with transversely isotropic symmetry. These results are used extensively throughout the body of the paper in the development of effective stress/strain relations for nonlinear laminated composites, which are a special class of transversely isotropic materials. The connection between the linear and nonlinear laminated composites is provided by the new variational principle in terms of the exact relation (3.3). The emphasis of this section is on representations for the transversely isotropic invariants of the stress and strain tensor. The reason is that *nonlinear* transversely isotropic materials are most efficiently characterized in terms of energy-density functions depending on these invariants.

I.1 *Isotropic invariants*

As is well-known, there are three isotropic invariants for a symmetric, second-order tensor. However, only two of these—those that are of quadratic order, or less—are relevant to linear-elastic behavior. These invariants may be expressed (see, for example, Walpole 1981) in terms of two fourth-order *projection* tensors $\mathbf{J}$ and $\mathbf{K}$, such that $\mathbf{I} = \mathbf{J} + \mathbf{K}$, $\mathbf{JJ} = \mathbf{J}$, $\mathbf{KK} = \mathbf{K}$ and $\mathbf{JK} = \mathbf{0}$. Their Cartesian components are given by

$$J_{ijkl} = \tfrac{1}{3}\delta_{ij}\delta_{kl}, \qquad K_{ijkl} = \tfrac{1}{2}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \tfrac{2}{3}\delta_{ij}\delta_{kl}\right), \tag{I.1}$$

where δ_{ij} is the Kronecker delta symbol. Then, in terms of these projection tensors, we define two isotropic invariants of the stress tensor via

$$\sigma_m = \tfrac{1}{3}J_{ktij}\sigma_{ij} \quad \text{and} \quad \tau_e^2 = \tfrac{1}{2}K_{ijkl}\sigma_{ij}\sigma_{kl}, \tag{I.2}$$

called the hydrostatic (mean) stress, and the effective shear stress, respectively. We also define the hydrostatic strain ε_m, and the effective shear strain γ_e by relations completely analogous to (I.2).

It is important to note that the elasticity tensor $\mathbf{L}$ of an isotropic, linear-elastic material admits a spectral decomposition

$$\mathbf{L} = 3\kappa\mathbf{J} + 2\mu\mathbf{K}, \tag{I.3}$$

where **J** and **K** play the role of the eigenprojections, and the bulk and shear moduli of the material, κ and μ, are the corresponding eigenvalues. As we will see next, the situation for transversely isotropic materials is different.

II.2 *Transversely isotropic invariants*

There are in general five transversely isotropic invariants of a symmetric, second-order tensor (Spencer 1971). However, only four of these invariants are linear, or quadratic, in order. They may be represented in terms of the four projections tensors (see Walpole 1981) $\mathbf{E}^{[1]}$, $\mathbf{E}^{[2]}$, $\mathbf{E}^{[3]}$, and $\mathbf{E}^{[4]}$, satisfying the relations $\mathbf{E}^{[p]}\mathbf{E}^{[p]} = \mathbf{E}^{[p]}$; $\mathbf{E}^{[p]}\mathbf{E}^{[q]} = \mathbf{0}, p \neq q$; and $\mathbf{E}^{[1]} + \mathbf{E}^{[2]} + \mathbf{E}^{[3]} + \mathbf{E}^{[4]} = \mathbf{I}$. The components of these four projections tensors are given respectively by

(I.4)
$$
\begin{aligned}
E^{[1]}_{ijkl} &= \tfrac{1}{2}\beta_{ij}\beta_{kl}, \\
E^{[2]}_{ijkl} &= \alpha_{ij}\alpha_{kl}, \\
E^{[3]}_{ijkl} &= \tfrac{1}{2}\left(\beta_{ik}\beta_{jl} + \beta_{jk}\beta_{il} - \beta_{ij}\beta_{kl}\right), \\
E^{[4]}_{ijkl} &= \tfrac{1}{2}\left(\beta_{ik}\alpha_{jl} + \beta_{il}\alpha_{jk} + \beta_{jl}\alpha_{ik} + \beta_{jk}\alpha_{il}\right),
\end{aligned}
$$

where $\alpha_{ij} = n_i n_j$ and $\beta_{ij} = \delta_{ij} - n_i n_j$, with $\mathbf{n}$ denoting the axis of transverse isotropy. Then, the four transversely isotropic invariants of the stress tensor $\boldsymbol{\sigma}$ may be expressed in the forms

(I.5)
$$
\begin{aligned}
\sigma_p &= \tfrac{1}{2}E^{[1]}_{iikl}\sigma_{kl} = \tfrac{1}{2}\sigma_{ij}\beta_{ij}, && \left\{ \tfrac{1}{2}(\sigma_{11} + \sigma_{22}) \right\}, \\
\sigma_n &= E^{[2]}_{iikl}\sigma_{kl} = \sigma_{ij}\alpha_{ij}, && \left\{ \sigma_{33} \right\}, \\
\tau_p &= \tfrac{1}{2}\sigma_{ij}E^{[3]}_{ijkl}\sigma_{kl} = \tfrac{1}{2}\left[\sigma_{ij}\sigma_{kl}\beta_{ik}\beta_{lj} - \tfrac{1}{2}\left(\sigma_{ij}\beta_{ij}\right)^2\right], && \left\{ \sigma_{12}^2 + \tfrac{1}{4}(\sigma_{11} - \sigma_{22})^2 \right\}. \\
\tau_n &= \tfrac{1}{2}\sigma_{ij}E^{[4]}_{ijkl}\sigma_{kl} = \left[\sigma_{ij}\sigma_{ki}\alpha_{jk} - \left(\sigma_{ij}\alpha_{ij}\right)^2\right], && \left\{ \left(\sigma_{13}^2 + \sigma_{23}^2\right) \right\},
\end{aligned}
$$

which correspond physically to the in-plane hydrostatic stress, the normal tensile stress, the (in-plane) transverse shear stress, and the (anti-plane) longitudinal shear stress (given in brackets are the corresponding representations for a choice of $\mathbf{n}$ aligned with the 3-direction). Analogous relations apply for the transversely isotropic invariants of the strain tensor $\boldsymbol{\varepsilon}$, denoted respectively ε_p, ε_n, γ_p, and γ_n. We also note for latter reference that the following two relations hold between the transversely isotropic invariants of (I.5) and the isotropic invariants of (I.2), namely,

(I.6) $$\sigma_m = \tfrac{1}{3}\left(2\sigma_p + \sigma_n\right), \qquad \tau_e^2 = \tau_p^2 + \tau_n^2 + \tfrac{1}{3}\left(\sigma_p - \sigma_n\right)^2.$$

Contrary to the situation for isotropic materials, the above four projection tensors are not the eigentensors of the spectral decomposition of an arbitrary transversely isotropic material (Mehrabadi and Cowin 1990). Such eigentensors would unfortunately involve the material moduli. Therefore, it is necessary to introduce (see Walpole 1981) two other tensors, that are *not* projections, $\mathbf{E}^{[5]}$ and $\mathbf{E}^{[6]}$, with components

(I.7) $$E^{[5]}_{ijkl} = \alpha_{ij}\beta_{kl},$$
$$E^{[6]}_{ijkl} = \beta_{ij}\alpha_{kl}.$$

Then, the elasticity tensor $\mathbf{L}$ of an arbitrary transversely isotropic material may be expressed in terms of these six tensors. It is worth mentioning that the above tensors satisfy the relation

(I.8) $$\mathbf{J} = \tfrac{2}{3}\mathbf{E}^{[1]} + \tfrac{1}{3}\mathbf{E}^{[2]} + \tfrac{1}{3}\left(\mathbf{E}^{[5]} + \mathbf{E}^{[6]}\right),$$

and that we can additionally define for latter reference an additional tensor $\mathbf{E}'$ such that

(I.9) $$\mathbf{E}' = \mathbf{E}^{[3]} + \mathbf{E}^{[4]} - \mathbf{K}.$$

This last tensor is a projection tensor, which is orthogonal to $\mathbf{E}^{[3]}$ and $\mathbf{E}^{[4]}$.

Finally, we remark that the energy density function of a transversely isotropic, linear-elastic materials may be represented in the form

(I.10) $$U(\boldsymbol{\sigma}) = \psi\left(\sigma_p, \sigma_n, \tau_p, \tau_n\right).$$

Then, the relation between the transversely isotropic stress and strain invariants is given by

(I.11) $$\varepsilon_p = \frac{1}{2}\frac{\partial \psi}{\partial \sigma_p}, \; \varepsilon_n = \frac{\partial \psi}{\partial \sigma_n}, \; \gamma_p = \frac{1}{2}\frac{\partial \psi}{\partial \tau_p}, \text{ and } \gamma_n = \frac{1}{2}\frac{\partial \psi}{\partial \tau_n}.$$

I.3 *Incompressible, transversely isotropic invariants*

For incompressible, transversely isotropic materials, it suffices to consider the three invariants of order less than quadratic on the space of traceless, symmetric, second-order tensors. These may be obtained in terms of the three orthogonal projection tensors $\mathbf{E}^{[3]}$, $\mathbf{E}^{[4]}$ and $\mathbf{E}'$, defined

in the previous subsection. Thus, the incompressible, transversely isotropic invariants of the stress tensor $\boldsymbol{\sigma}$ are τ_p, τ_n, and the deviatoric shear stress

$$\tau_d = \frac{1}{\sqrt{3}}\left(\sigma_p - \sigma_n\right), \tag{I.12}$$

corresponding to the three above projections, respectively. We note further that from $(I.6)_2$ we have the following identity relating the effective shear stress and the incompressible, transversely isotropic invariants, $\tau_e^2 = \tau_p^2 + \tau_n^2 + \tau_d^2$. The corresponding strain invariants are, of course, denoted γ_p, γ_n and γ_d.

Finally, we note that the elasticity tensor $\mathbf{L}$ of an incompressible, transversely isotropic, linear-elastic material admits a spectral decomposition of the form

$$\mathbf{L} = 2\mu_p \mathbf{E}^{[3]} + 2\mu_n \mathbf{E}^{[4]} + 2\mu_d \mathbf{E}^{'}, \tag{I.13}$$

where μ_p, μ_n, μ_d, are the three shear moduli that suffice to characterize the behavior of such a material (Lipton, 1991a).

Appendix II: A useful identity

In this appendix, we demonstrate the following identity

$$\frac{1}{\overline{\alpha}} = \min_{\omega^{(r)}, \overline{\omega}=1} \left\{ \sum_{r=1}^{n} \frac{c^{(r)}}{\alpha^{(r)}} \left(\omega^{(r)}\right)^2 \right\}, \tag{II.1}$$

which is used repeatedly in the body of the paper.

We begin by letting g be the function defined by

$$g\left(\omega^{(r)}\right) = \sum_{r=1}^{n} \frac{c^{(r)}}{\alpha^{(r)}} \left(\omega^{(r)}\right)^2, \tag{II.2}$$

where the variables $\omega^{(r)}$ (r = 1,..., n) are subject to the constraint $\overline{\omega} = 1$, and where the variables $\alpha^{(r)} > 0$ (r = 1,..., n).

The choice of the set, $\omega^{(r)} = \alpha^{(r)}/\overline{\alpha}$, satisfies the constraint and is such that $g\left(\omega^{(r)}\right) = 1/\overline{\alpha}$. Consider next a second, arbitrary set, distinct from the first set, $\hat{\omega}^{(r)}$ $(r = 1,..., n)$, such that $\overline{\hat{\omega}} = 1$, and let $\theta^{(r)} = \hat{\omega}^{(r)} - \omega^{(r)}$. Then, substitution of this second set into (II.2) leads to

$$(II.2) \qquad g\left(\hat{\omega}^{(r)}\right) = \sum_{r=1}^{n} \frac{c^{(r)}}{\alpha^{(r)}}\left(\hat{\omega}^{(r)}\right)^2 = \sum_{r=1}^{n} \frac{c^{(r)}}{\alpha^{(r)}}\left(\omega^{(r)}\right)^2 + \sum_{r=1}^{n} \frac{c^{(r)}}{\alpha^{(r)}}\left(\theta^{(r)}\right)^2 > g\left(\omega^{(r)}\right),$$

where we have used the fact that $\overline{\theta} = 0$. Hence, identity (I.1) is demonstrated. In the body of the paper, we replace $\omega^{(r)}$ by $(1 - \omega^{(r)})$, with an appropriate modification for the constraint.

Appendix III: A simplified expression for the effective stress/strain relations

Consider the following form for the effective energy function (4.8) of the incompressible laminated composite

$$(III.1) \qquad \tilde{U}(\overline{\sigma}) = \min_{\omega^{(r)},\, \overline{\omega}=0} \left\{ \sum_{s=1}^{n} c^{(s)} \psi^{(s)}\left(\tau^{(s)}\right) \right\},$$

where $\tau^{(s)} = \sqrt{\left(\overline{\tau}_p^2 + \overline{\tau}_d^2\right)\left(1 - \omega^{(s)}\right)^2 + \overline{\tau}_n^2}$. As shown in the body of the paper, we can eliminate the constraint $\overline{\omega} = 0$ by letting

$$(III.2) \qquad \omega^{(n)} = -\frac{1}{c^{(n)}} \sum_{r=1}^{n-1} c^{(r)} \omega^{(r)},$$

and rewriting (III.1) in terms of the $n-1$ optimizations variables $\omega^{(r)}$ $(r = 1,..., n-1)$ via

$$(III.3) \qquad \tilde{U}(\overline{\sigma}) = \min_{\substack{\omega^{(r)} \\ r=1,...,n-1}} \left\{ \sum_{s=1}^{n-1} c^{(s)} \psi^{(s)}\left(\tau^{(s)}\right) + c^{(n)} \psi^{(n)}\left(\tau^{(n)}\right) \right\},$$

where the variables $\tau^{(s)}$ $(s = 1,..., n-1)$ are the same as before, but on the other hand $\tau^{(n)} = \sqrt{\left(\overline{\tau}_p^2 + \overline{\tau}_d^2\right)\left(1 + \frac{1}{c^{(n)}} \sum_{t=1}^{n-1} c^{(t)} \omega^{(t)}\right)^2 + \overline{\tau}_n^2}$. Then, the $n-1$ optimization conditions of (III.3) are given by the relations

$$(III.4) \qquad -\frac{1}{\tau^{(r)}}\left(\psi^{(r)}\right)'\left(\tau^{(r)}\right)\left(1 - \omega^{(r)}\right) + \frac{1}{\tau^{(n)}}\left(\psi^{(n)}\right)'\left(\tau^{(n)}\right)\left(1 + \frac{1}{c^{(n)}} \sum_{s=1}^{n-1} c^{(s)} \omega^{(s)}\right) = 0, \ (r = 1,..., n-1).$$

If we now denote the optimal variables $\omega^{(r)}$, satisfying (III.4), by $\hat{\omega}^{(r)}$ $(r=1,..,n-1)$, the effective energy function of the incompressible laminate may then be written in the form

$$\tilde{U}(\overline{\sigma}) = \sum_{s=1}^{n-1} c^{(s)} \psi^{(s)}\left(\hat{\tau}^{(s)}\right) + c^{(n)} \psi^{(n)}\left(\hat{\tau}^{(n)}\right), \tag{III.5}$$

where $\hat{\tau}^{(s)} = \tau^{(s)}\left(\omega^{(s)} = \hat{\omega}^{(s)}\right)$ and $\hat{\tau}^{(n)} = \sqrt{\left(\overline{\tau}_p^2 + \overline{\tau}_d^2\right)\left(1 + \frac{1}{c^{(n)}} \sum_{t=1}^{n-1} c^{(t)} \hat{\omega}^{(t)}\right)^2 + \overline{\tau}_n^2}$.

It follows that effective stress/strain relations of the laminated composite may be computed from the relations

$$\begin{aligned} \overline{\varepsilon} = & \sum_{r=1}^{n-1} \frac{c^{(r)}}{\hat{\tau}^{(r)}} \left(\psi^{(r)}\right)' \left(\hat{\tau}^{(r)}\right) \left[\left(1 - \hat{\omega}^{(r)}\right)^2 \left(\overline{\tau}_p \frac{\partial \overline{\tau}_p}{\partial \overline{\sigma}} + \overline{\tau}_d \frac{\partial \overline{\tau}_d}{\partial \overline{\sigma}} \right) + \overline{\tau}_n \frac{\partial \overline{\tau}_n}{\partial \overline{\sigma}} \right] + \ldots \\ & \ldots + \frac{c^{(n)}}{\hat{\tau}^{(n)}} \left(\psi^{(n)}\right)' \left(\hat{\tau}^{(n)}\right) \left[\left(1 + \frac{1}{c^{(n)}} \sum_{s=1}^{n-1} c^{(s)} \hat{\omega}^{(s)}\right)^2 \left(\overline{\tau}_p \frac{\partial \overline{\tau}_p}{\partial \overline{\sigma}} + \overline{\tau}_d \frac{\partial \overline{\tau}_d}{\partial \overline{\sigma}} \right) + \overline{\tau}_n \frac{\partial \overline{\tau}_n}{\partial \overline{\sigma}} \right] + \ldots \\ & \ldots + \sum_{r=1}^{n-1} c^{(r)} \left(\overline{\tau}_p^2 + \overline{\tau}_d^2\right) \frac{\partial \hat{\omega}^{(r)}}{\partial \overline{\sigma}} \left[-\frac{1}{\hat{\tau}^{(r)}} \left(\psi^{(r)}\right)' \left(\hat{\tau}^{(r)}\right) \left(1 - \hat{\omega}^{(r)}\right) + \frac{1}{\hat{\tau}^{(n)}} \left(\psi^{(n)}\right)' \left(\hat{\tau}^{(n)}\right) \left(1 + \frac{1}{c^{(n)}} \sum_{s=1}^{n-1} c^{(s)} \hat{\omega}^{(s)}\right) \right]. \end{aligned} \tag{III.6}$$

We note that each of the terms in the last summation of (III.6) is identical to zero by virtue of the optimizations conditions (III.4). Thus, in the computation of the effective stress/strain relations, we may regard the optimizations variables as constants as far as derivatives with respect to $\overline{\sigma}$ are concerned, to obtain the final result

$$\overline{\varepsilon} = \sum_{r=1}^{n} \frac{c^{(r)}}{\hat{\tau}^{(r)}} \left(\psi^{(r)}\right)' \left(\hat{\tau}^{(r)}\right) \left[\left(1 - \hat{\omega}^{(r)}\right)^2 \left(\overline{\tau}_p \frac{\partial \overline{\tau}_p}{\partial \overline{\sigma}} + \overline{\tau}_d \frac{\partial \overline{\tau}_d}{\partial \overline{\sigma}} \right) + \overline{\tau}_n \frac{\partial \overline{\tau}_n}{\partial \overline{\sigma}} \right], \tag{III.7}$$

where $\hat{\omega}^{(n)}$ is defined via the relation (III.2) in terms of the other $\hat{\omega}^{(r)}$ $(r = 1,..., n-1)$.

It can be shown that an analogous result may be obtained for the nonlinear *compressible* composite with effective energy function $\tilde{U}$ given by (5.8). In fact, we may write the effective stress/strain relations for the nonlinear compressible laminate in the form

$$\overline{\varepsilon} = \sum_{r=1}^{n} c^{(r)} \frac{\partial \psi^{(r)}}{\partial \overline{\sigma}} \left(\hat{\tau}_e^{(r)}, \hat{\sigma}_m^{(r)}\right), \tag{III.8}$$

where $\hat{\tau}_c^{(r)}$, and $\hat{\sigma}_m^{(r)}$ are evaluated from (5.7) at the optimal values of $\omega_c^{(r)}$, and $\omega_m^{(r)}$, denoted by $\hat{\omega}_c^{(r)}$, and $\hat{\omega}_m^{(r)}$, respectively. Here, the derivatives with respect to the average stress $\overline{\sigma}$ are evaluated with $\hat{\omega}_c^{(r)}$, and $\hat{\omega}_m^{(r)}$ fixed.

Appendix IV: More on the compressible laminated composites

Having obtained expressions (5.6) and (5.8) for the effective energy functions of the linear and nonlinear laminated composites, $\tilde{U}_o$ and $\tilde{U}$, we note that the form of these expressions is reminiscent of the type of result that one would expect from direct utilization of the principle of minimum complementary energy (2.4). In this appendix, we briefly show that results (5.6) and (5.8) for $\tilde{U}_o$ and $\tilde{U}$, respectively, can indeed be alternatively obtained directly from the principle of minimum complementary energy. It is important to emphasize, however, that while the derivations given in the body of the paper result from straightforward mechanical computations, the present derivations based on the principle of minimum complementary energy rely on the physics of the problem, and were *motivated* by the prior derivations (*i.e.*, we would not have discovered them had they not been derived by the prior analysis). Also, the case of a laminated composite is a very special case of a composite material; in general, we do *not* expect that other types of results, such as bounds for the effective properties of nonlinear isotropic composites, obtained by means of the new variational principle (*e.g.* Ponte Castañeda 1991a, b, c), may also be obtained directly from the minimum complementary-energy principle.

We begin with the derivation of the linear result (5.6). We have already mentioned that the stress field within the laminated composite is piecewise constant, *i.e.*, of the form $\boldsymbol{\sigma} = \sum_{r=1}^{n} \chi^{(r)}(\mathbf{x}\cdot\mathbf{n})\boldsymbol{\sigma}^{(r)}$, where $\boldsymbol{\sigma}^{(r)}$ corresponds to the constant stress field in phase r. The problem then reduces to that of finding these unknown phase stresses $\boldsymbol{\sigma}^{(r)}$, together with the corresponding constant strain fields $\boldsymbol{\varepsilon}^{(r)}$ (related to the stresses by the phase constitutive relations), and satisfying

the conditions of continuity of the traction stresses and tangential strains across the interfaces between the phases, as well as the averaging conditions stated in § 2.

In this connection, the interior and exterior projection operators of Hill (1972, 1983) $\mathbf{F} = \mathbf{E}^{[1]} + \mathbf{E}^{[3]}$ and $\mathbf{E} = \mathbf{E}^{[2]} + \mathbf{E}^{[4]}$ (refer to Appendix I), respectively, turn out to be useful because they allow the decomposition of any symmetric, second-order tensor into its tangential (interior) and traction (exterior) components (with reference to a boundary with normal $\mathbf{n}$). Thus, the tangential components of the strain (which must be continuous across interphase boundaries on the laminated composite) are given by $\mathbf{F}\boldsymbol{\varepsilon}$, and, correspondingly, the traction components of the stress (which must also be continuous across the interphase boundaries) are given by $\mathbf{E}\boldsymbol{\sigma}$. Alternatively, we may state that $\mathbf{E}^{[2]}\boldsymbol{\sigma}$, $\mathbf{E}^{[4]}\boldsymbol{\sigma}$ and $\mathbf{E}^{[1]}\boldsymbol{\varepsilon}$, $\mathbf{E}^{[3]}\boldsymbol{\varepsilon}$ must also be continuous across such boundaries.

Next, we apply the above results to the laminated composite, for which the interfacial boundaries are all perpendicular to a fixed vector $\mathbf{n}$. Since the traction stresses must be continuous from phase to phase, we have that

(IV.1) $$\mathbf{E}^{[2]}\boldsymbol{\sigma}^{(r)} = \mathbf{E}^{[2]}\overline{\boldsymbol{\sigma}}, \text{ and } \mathbf{E}^{[4]}\boldsymbol{\sigma}^{(r)} = \mathbf{E}^{[4]}\overline{\boldsymbol{\sigma}},$$

where we have additionally made use of the average stress condition given in § 2. We continue by noting that for an isotropic material (as are all the phases in the our laminate), $\mathbf{E}^{[3]}\boldsymbol{\sigma}^{(r)} = 2\mu_o^{(r)}\mathbf{E}^{[3]}\boldsymbol{\varepsilon}^{(r)}$ within each linear phase, and therefore for an isotropic phase the $\mathbf{E}^{[3]}$ projection of the stress tensor must have the same direction in all phases. Thus, applying the averaging condition for the stresses, we arrive at

(IV.2) $$\mathbf{E}^{[3]}\boldsymbol{\sigma}^{(r)} = \left(1 - \omega_e^{(r)}\right)\mathbf{E}^{[3]}\overline{\boldsymbol{\sigma}},$$

where the variables $\omega_e^{(r)}$ must satisfy the condition that $\overline{\omega}_e = 0$. Additionally, since the $\mathbf{E}^{[1]}$-projection is one-dimensional, it follows that the $\mathbf{E}^{[1]}$-projections of the stress tensor must also be parallel from phase to phase. Therefore, applying the averaging condition for the stresses, we have that

(IV.3) $$\mathbf{E}^{[1]}\boldsymbol{\sigma}^{(r)} = \left(1 - \omega_m^{(r)}\right)\mathbf{E}^{[1]}\overline{\boldsymbol{\sigma}},$$

where the variables $\omega_m^{(r)}$ must satisfy the condition that $\overline{\omega}_m = 0$. We note, however, that if $\mathbf{E}^{[1]}\overline{\boldsymbol{\sigma}} = 0$ (or, equivalently, if $\overline{\sigma}_p = 0$), the above result does not hold, because in this case the corresponding projections of the stress in the phases need not vanish (only their average needs to vanish).

Applying the results of Appendix I [in particular, (I.6)], we conclude that the isotropic invariants of the stress tensor within each phase $\tau_e^{(r)}$ and $\sigma_m^{(r)}$ (on which the energy-density functions of each isotropic phase depend) are precisely those given by relations (5.7). Therefore, it follows from the principle of minimum complementary energy—by minimizing over the set of admissible stresses (*i.e.*, over the optimizing variables $\omega_e^{(r)}$ and $\omega_m^{(r)}$ subject to the constraints $\overline{\omega}_e = 0$ and $\overline{\omega}_m = 0$)—that the effective energy function $\tilde{U}_o$ of the linear composite is indeed given by expression (5.6).

For the nonlinear laminated composite, we observe that the same analysis given above would also work, leading to expression (5.8) for $\tilde{U}$. The only modification that is required in this analysis is that for a nonlinear isotropic phase (say phase r), the relation $\mathbf{E}^{[3]}\boldsymbol{\sigma}^{(r)} = 2\mu_o^{(r)}\mathbf{E}^{[3]}\boldsymbol{\varepsilon}^{(r)}$ would not hold, but it can be easily shown that for the nonlinear isotropic material of the type considered in this work, the conclusion (IV.2) would still hold, and hence the final form for $\tilde{U}$ would be the same as that for the linear laminated composite $\tilde{U}_o$.

We conclude this appendix by stating an alternative form of (5.6) and (5.8) that works even when $\overline{\sigma}_p = 0$). This is accomplished by redefining the optimizing variables $\omega_m^{(r)}$ in terms of the new variables

$$\hat{\omega}_m^{(r)} = \frac{\overline{\sigma}_p \omega_m^{(r)} - \overline{\sigma}_n}{\overline{\sigma}_p - \overline{\sigma}_n}, \tag{IV.4}$$

where now we need to have that $\overline{\sigma}_p - \overline{\sigma}_n \neq 0$. In terms of the new variables $\hat{\omega}_m^{(r)}$ (the variables $\omega_e^{(r)}$ do not change), relation (5.8) is expressed in the form

$$\tilde{U}(\overline{\boldsymbol{\sigma}}) = \min_{\substack{\omega_e^{(r)},\,\hat{\omega}_m^{(r)} \\ \overline{\omega}_e = \overline{\hat{\omega}}_m = 0}} \left\{ \sum_{s=1}^{n} c^{(s)} \psi^{(s)}\left(\hat{\tau}_e^{(s)}, \hat{\sigma}_m^{(s)}\right) \right\}, \tag{IV.5}$$

where

(IV.6) $\hat{\tau}_e^{(s)} = \sqrt{\left(1-\omega_e^{(s)}\right)^2 \overline{\tau}_p^2 + \overline{\tau}_n^2 + \frac{1}{3}\left[\left(\overline{\sigma}_n - \overline{\sigma}_p\right)\left(1-\hat{\omega}_m^{(s)}\right)\right]^2}$ and $\hat{\sigma}_m^{(s)} = \overline{\sigma}_n + \frac{2}{3}\left(\overline{\sigma}_p - \overline{\sigma}_n\right)\left(1-\hat{\omega}_m^{(s)}\right)$.

Note that when $\overline{\sigma}_p = \overline{\sigma}_n = 0$, we are guaranteed that $\sigma_p^{(r)} = \sigma_n^{(r)} = 0$ in each phase, and then both forms are equally valid.

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Fig. 1. A two-phase laminated material.

Fig. 2. The relations between the longitudinal shear stress $\bar{\tau}_n$ and strain $\bar{\gamma}_n$ of the incompressible, nonlinear laminate (continuous lines), and the reference linear laminate (short-dashed lines) for three different values of the other stress mode $\bar{\tau}$; $\bar{\tau}/\tau_0 = 0$, $\bar{\tau}/\tau_0 = 2$, and $\bar{\tau}/\tau_0 = 5$.

Fig. 3. The inter-relations between the shear stress $\bar{\tau}$ and the longitudinal shear strain $\bar{\gamma}_n$ of the incompressible, nonlinear laminate (continuous lines), and the reference linear laminate (short-dashed lines) for two different values of the longitudinal shear stress $\bar{\tau}_n$; $\bar{\tau}_n/\tau_0 = 0.5$, and $\bar{\tau}_n/\tau_0 = 1$.

Fig 4. The relations between the shear stress $\bar{\tau}$ and the corresponding shear strain $\bar{\gamma}$ of the incompressible, nonlinear laminate (continuous lines), and the reference linear laminate (short-dashed lines) for three different values of the longitudinal shear stress $\bar{\tau}_n$; $\bar{\tau}_n/\tau_0 = 0$, $\bar{\tau}_n/\tau_0 = 0.5$, and $\bar{\tau}_n/\tau_0 = 2$.

Fig. 5. The inter-relations between the longitudinal shear stress $\bar{\tau}_n$ and the shear strain $\bar{\gamma}$ of the incompressible, nonlinear laminate (continuous lines), and the reference linear laminate (short-dashed lines) for two different values of the stress mode $\bar{\tau}$; $\bar{\tau}/\tau_0 = 0.5$, and $\bar{\tau}/\tau_0 = 1$.

Fig. 6. The relations between the in-plane hydrostatic stress $\bar{\sigma}_p$ and strain $\bar{\varepsilon}_p$ of the compressible laminate (where no other stress modes are present) for three different values of the volume fraction of the linear phase, $c^{(2)}$.

Fig. 7. The relations between the normal tensile stress $\bar{\sigma}_n$ and strain $\bar{\varepsilon}_n$ of the compressible laminate (where no other stress modes are present) for three different values of the volume fraction of the linear phase, $c^{(2)}$.

Fig. 8. The relations between the transverse shear stress $\bar{\tau}_p$ and strain $\bar{\gamma}_p$ of the compressible laminate (where no other stress modes are present) for three different values of the volume fraction of the linear phase, $c^{(2)}$.

Fig. 9. The relations between the longitudinal shear stress $\bar{\tau}_n$ and strain $\bar{\gamma}_n$ of the compressible laminate (where no other stress modes are present) for three different values of the volume fraction of the linear phase, $c^{(2)}$.

Fig. 10. The exact yield surface of the laminated composite, the anisotropic elliptic estimate of Hill and the Voigt and Reuss isotropic estimates plotted on the $(\bar{\tau}_n, \bar{\tau})$-plane for $\tau_o^{(2)}/\tau_o^{(1)} = 2$ and $c^{(2)} = 0.5$.

Fig. 11. The exact yield surface of the laminated composite (continuous lines) and the quadratic approximation of Hill (dashed lines) plotted on the $(\bar{\tau}_n, \bar{\tau})$-plane for $\tau_o^{(2)}/\tau_o^{(1)} = 1.25$ and three values of $c^{(2)}$ (0.1, 0.5 and 0.9).

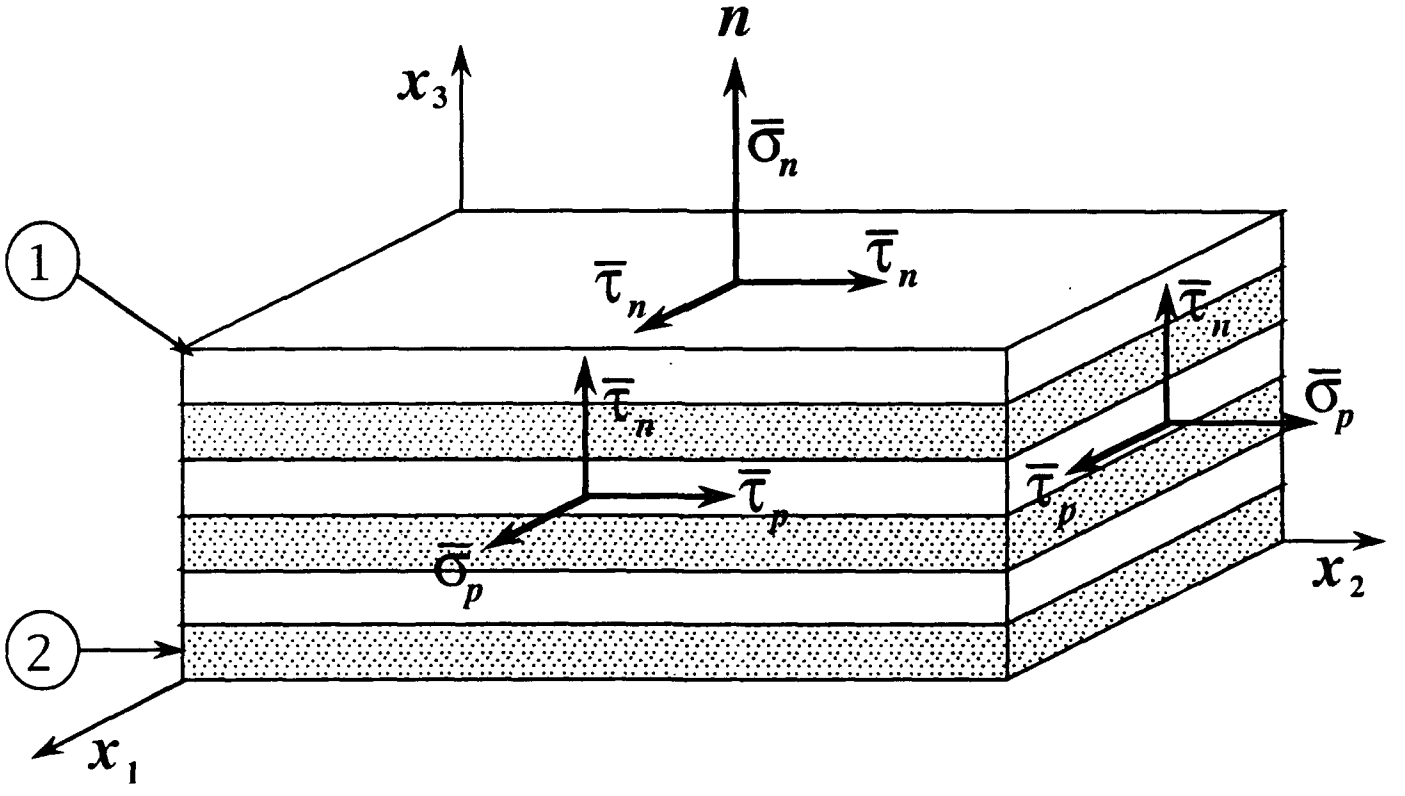

Fig-1

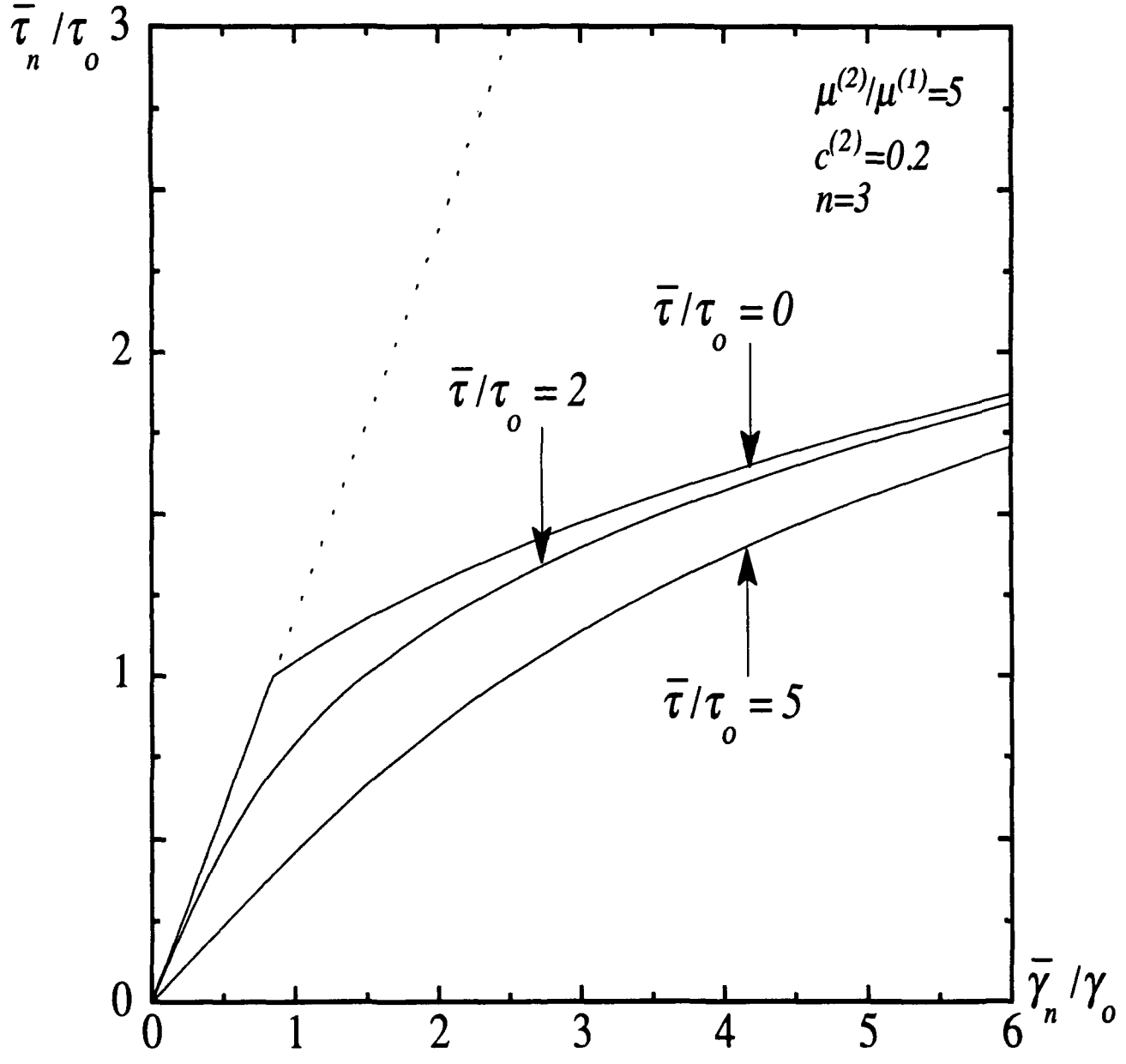
$\bar{\tau}_n/\tau_o$
3
2
1
0
0
1
2
3
4
5
6
$\bar{\gamma}_n/\gamma_o$
$\mu^{(2)}/\mu^{(1)}=5$
$c^{(2)}=0.2$
$n=3$
$\bar{\tau}/\tau_o=0$
$\bar{\tau}/\tau_o=2$
$\bar{\tau}/\tau_o=5$

fig. 2

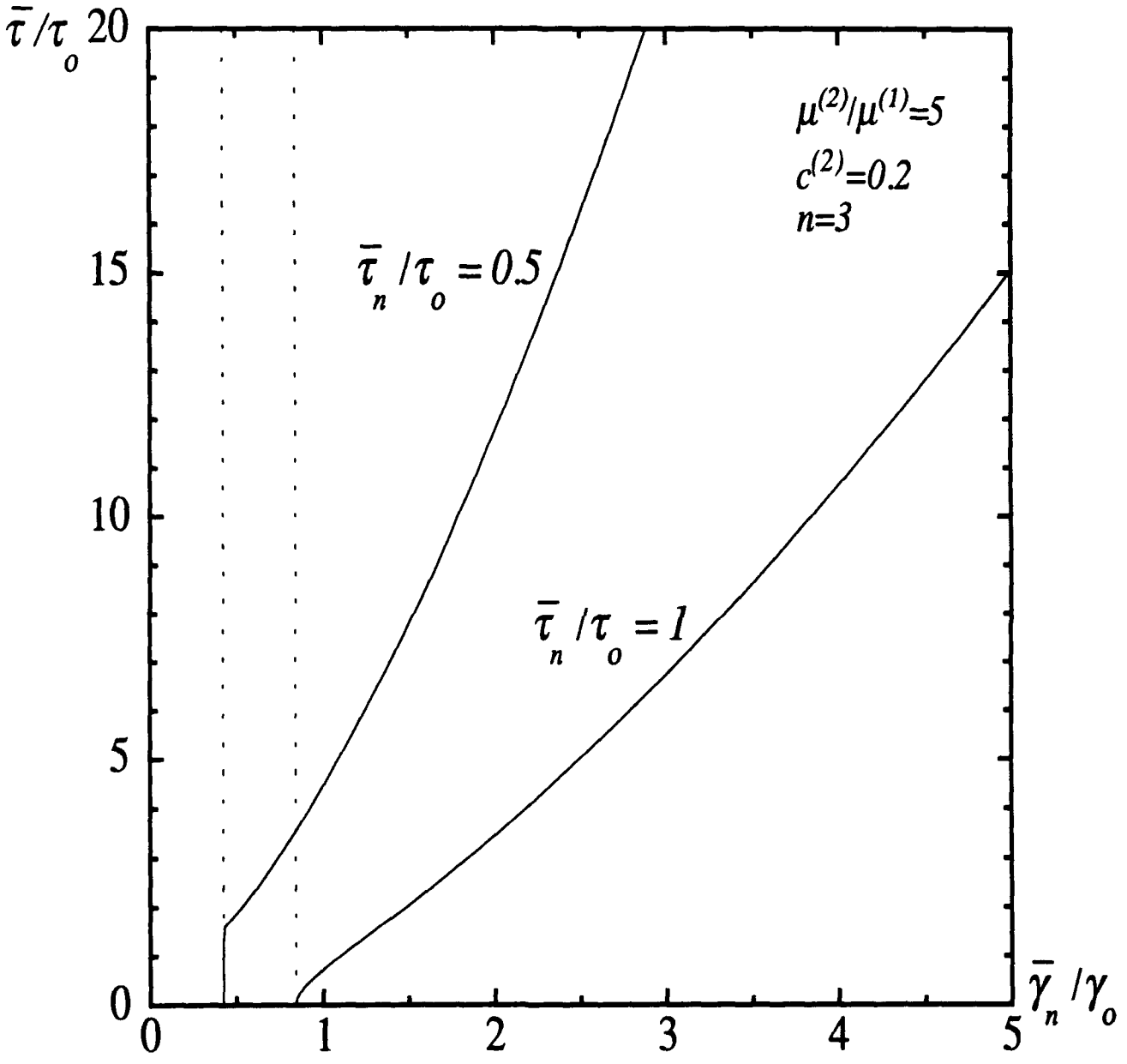
$\bar{\tau}/\tau_o$ 20
15
10
5
0
0
1
2
3
4
5
$\bar{\gamma}_n/\gamma_o$
$\mu^{(2)}/\mu^{(1)}=5$
$c^{(2)}=0.2$
$n=3$
$\bar{\tau}_n/\tau_o=0.5$
$\bar{\tau}_n/\tau_o=1$

fig-3

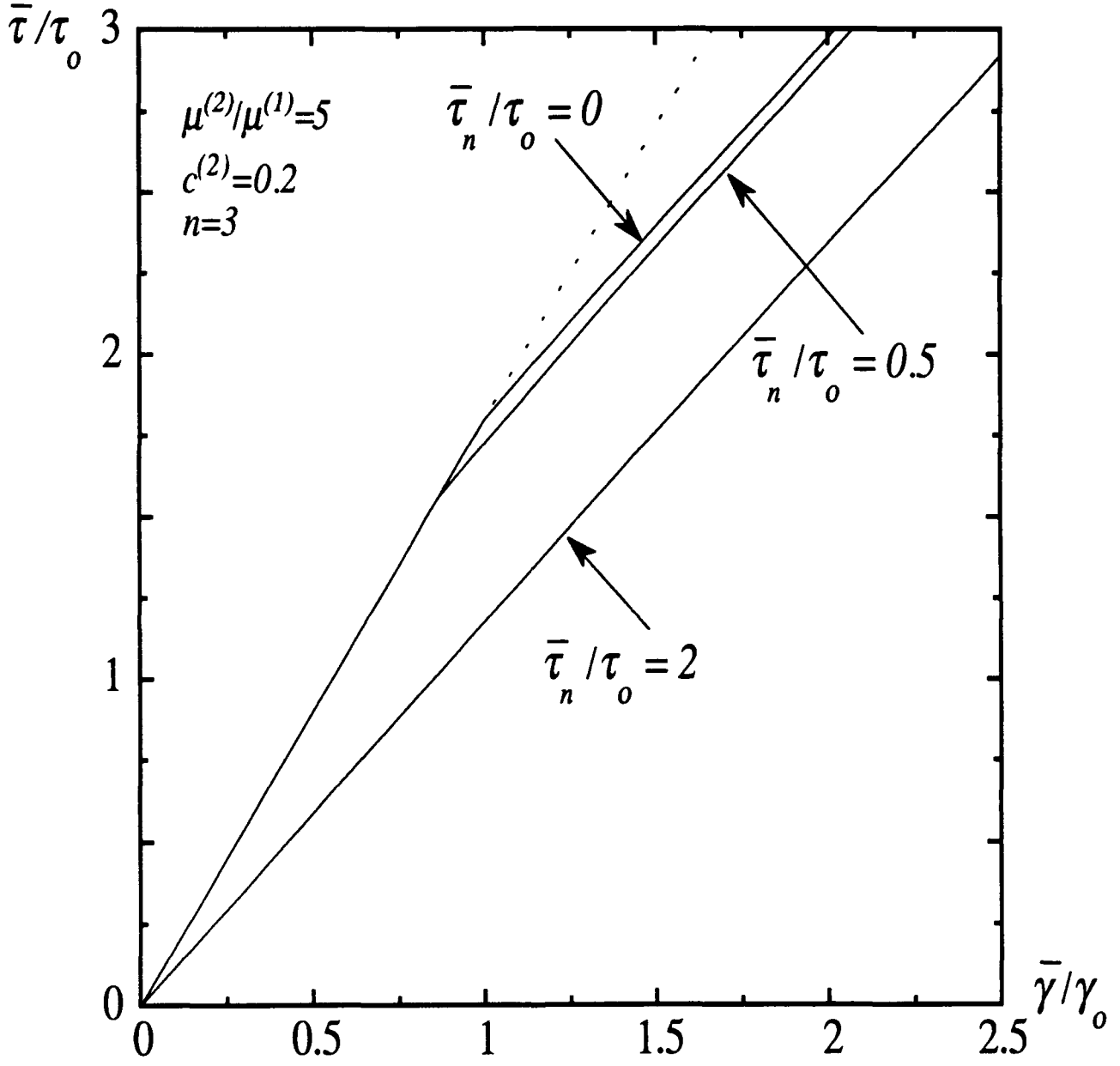
$\bar{\tau}/\tau_o$
3
2
1
0
$\mu^{(2)}/\mu^{(1)}=5$
$c^{(2)}=0.2$
$n=3$
$\bar{\tau}_n/\tau_o=0$
$\bar{\tau}_n/\tau_o=0.5$
$\bar{\tau}_n/\tau_o=2$
0
0.5
1
1.5
2
2.5
$\bar{\gamma}/\gamma_o$

fig - 4

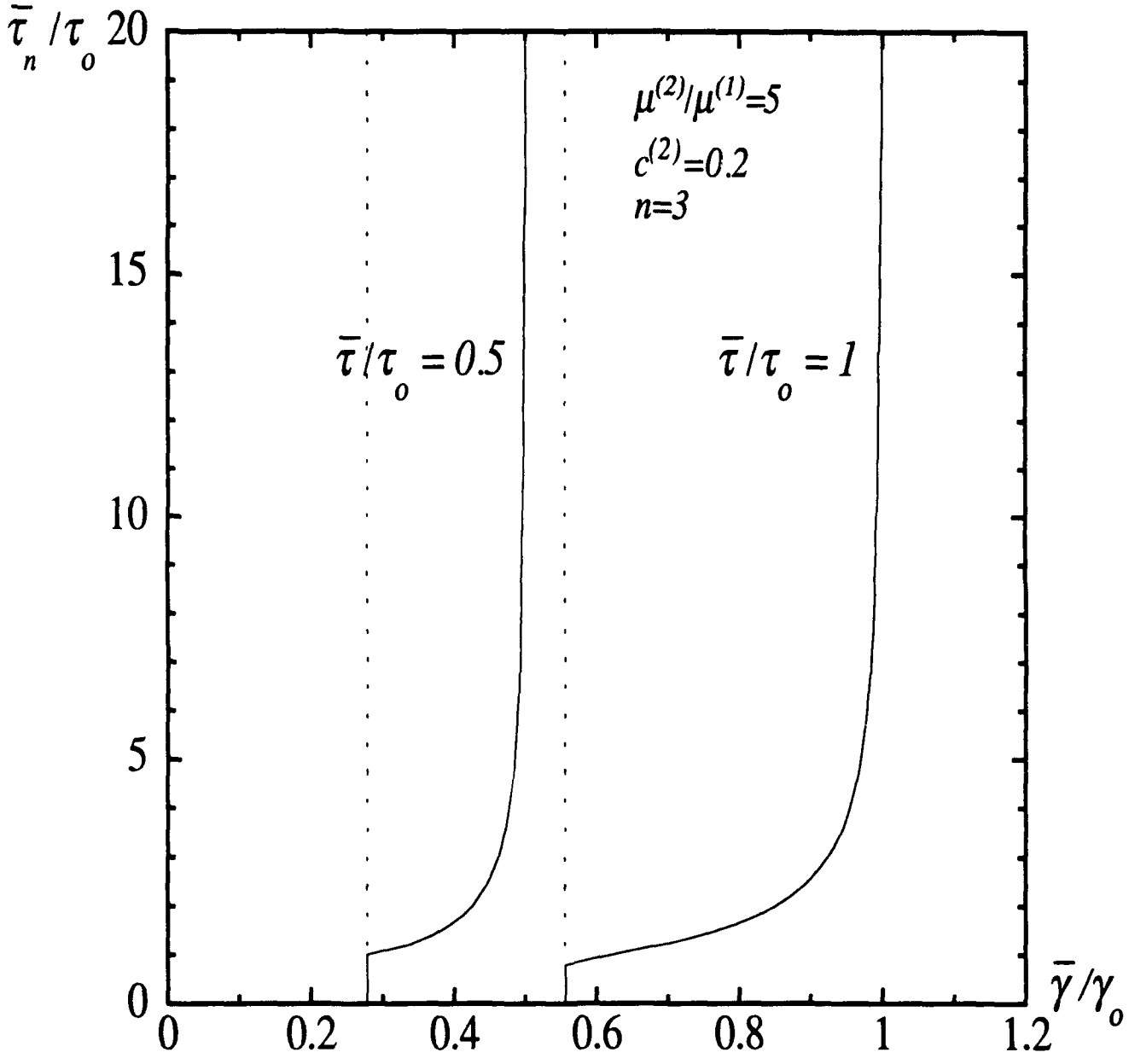
$\bar{\tau}_n/\tau_o$
20
15
10
5
0
0
0.2
0.4
0.6
0.8
1
1.2
$\bar{\gamma}/\gamma_o$
$\mu^{(2)}/\mu^{(1)}=5$
$c^{(2)}=0.2$
$n=3$
$\bar{\tau}/\tau_o=0.5$
$\bar{\tau}/\tau_o=1$

fig-5

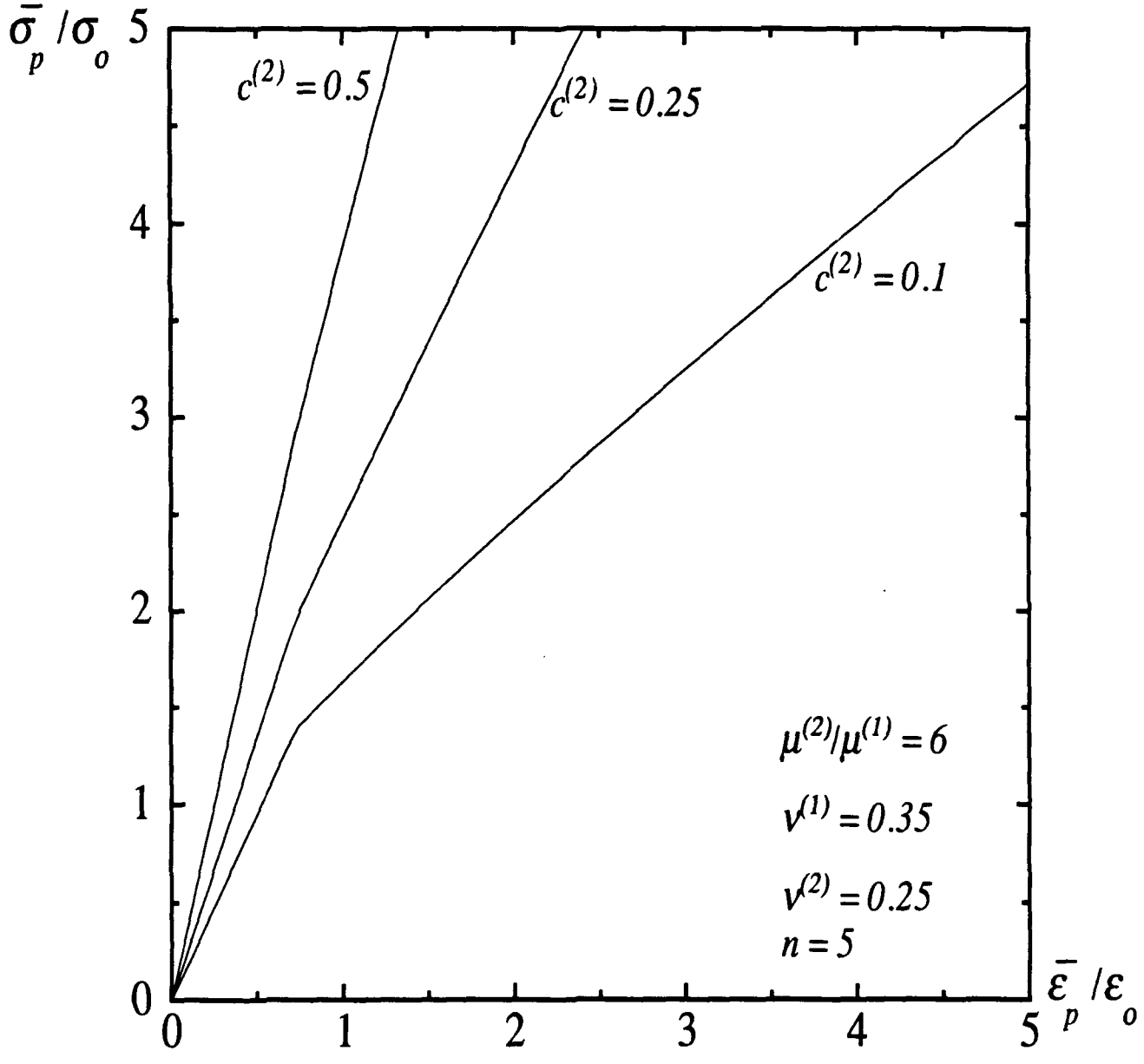
$\bar{\sigma}_p/\sigma_o$
5
4
3
2
1
0
0
1
2
3
4
5
$\bar{\varepsilon}_p/\varepsilon_o$
$c^{(2)}=0.5$
$c^{(2)}=0.25$
$c^{(2)}=0.1$
$\mu^{(2)}/\mu^{(1)}=6$
$\nu^{(1)}=0.35$
$\nu^{(2)}=0.25$
$n=5$

fig-6

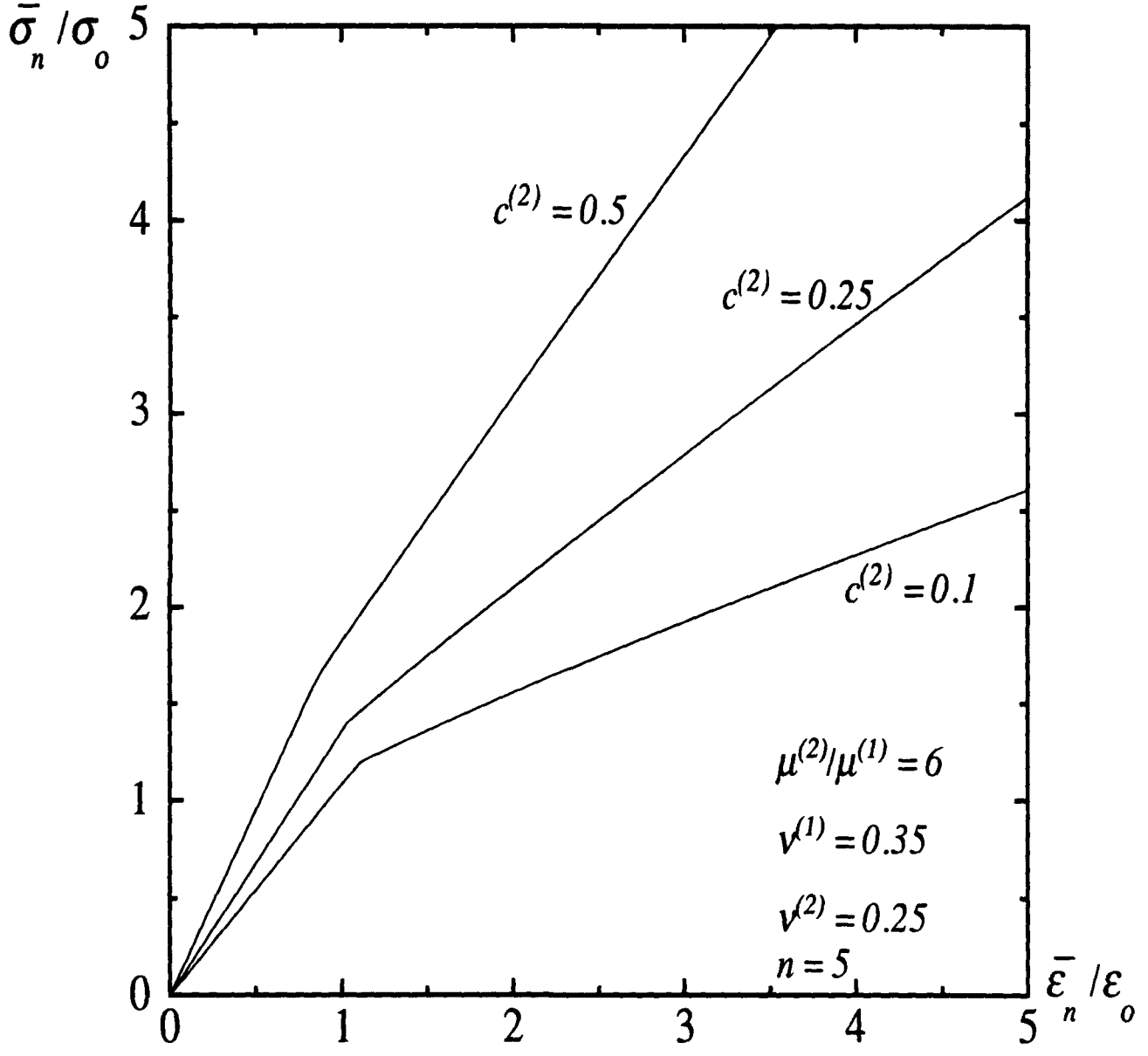
$\bar{\sigma}_n/\sigma_o$
5
4
3
2
1
0
0
1
2
3
4
5
$\bar{\varepsilon}_n/\varepsilon_o$
$c^{(2)}=0.5$
$c^{(2)}=0.25$
$c^{(2)}=0.1$
$\mu^{(2)}/\mu^{(1)}=6$
$\nu^{(1)}=0.35$
$\nu^{(2)}=0.25$
$n=5$

fig-7

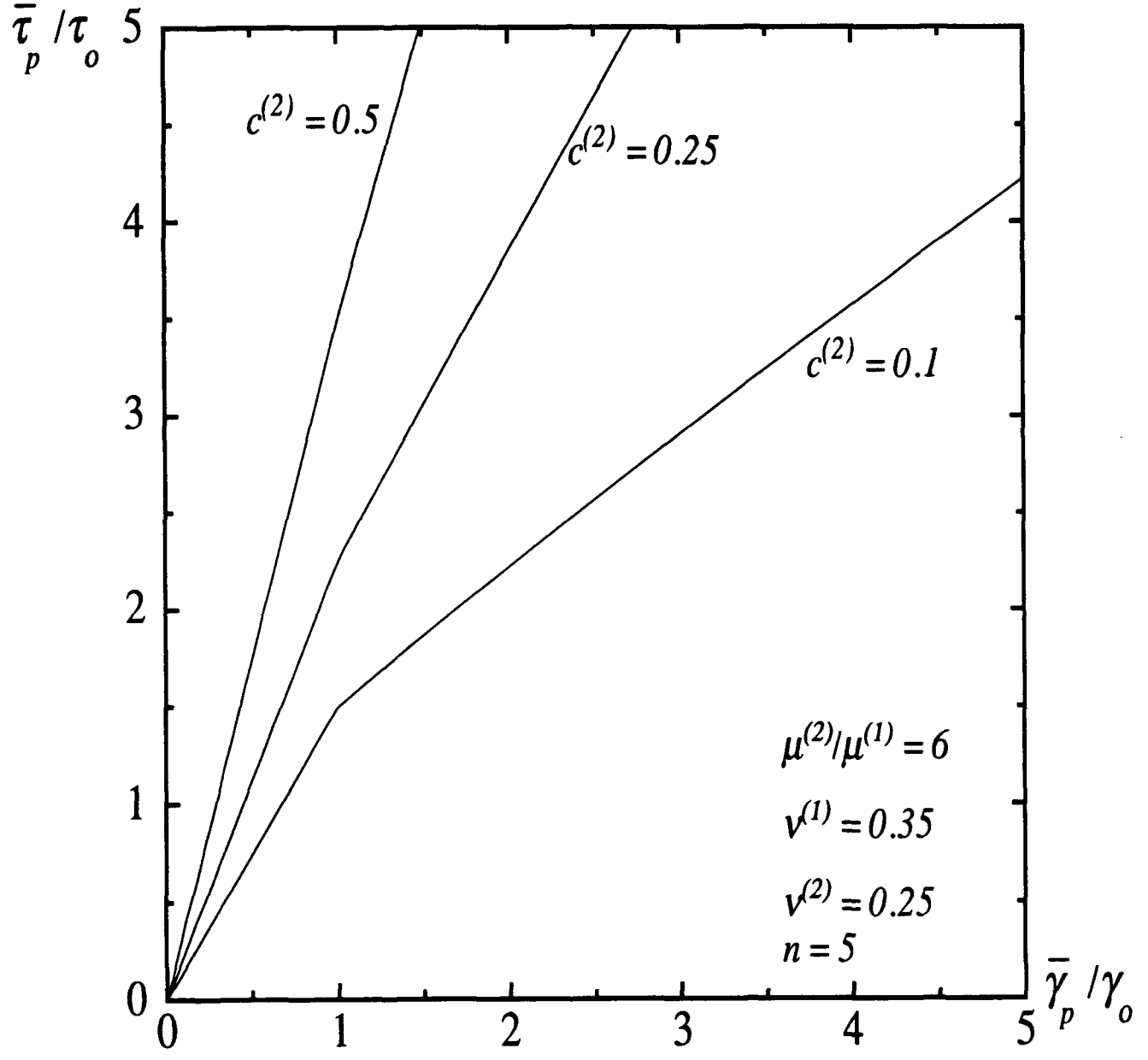
$\bar{\tau}_p / \tau_o$
5
4
3
2
1
0
0
1
2
3
4
5
$\bar{\gamma}_p / \gamma_o$
$c^{(2)} = 0.5$
$c^{(2)} = 0.25$
$c^{(2)} = 0.1$
$\mu^{(2)}/\mu^{(1)} = 6$
$\nu^{(1)} = 0.35$
$\nu^{(2)} = 0.25$
$n = 5$

fig - 8

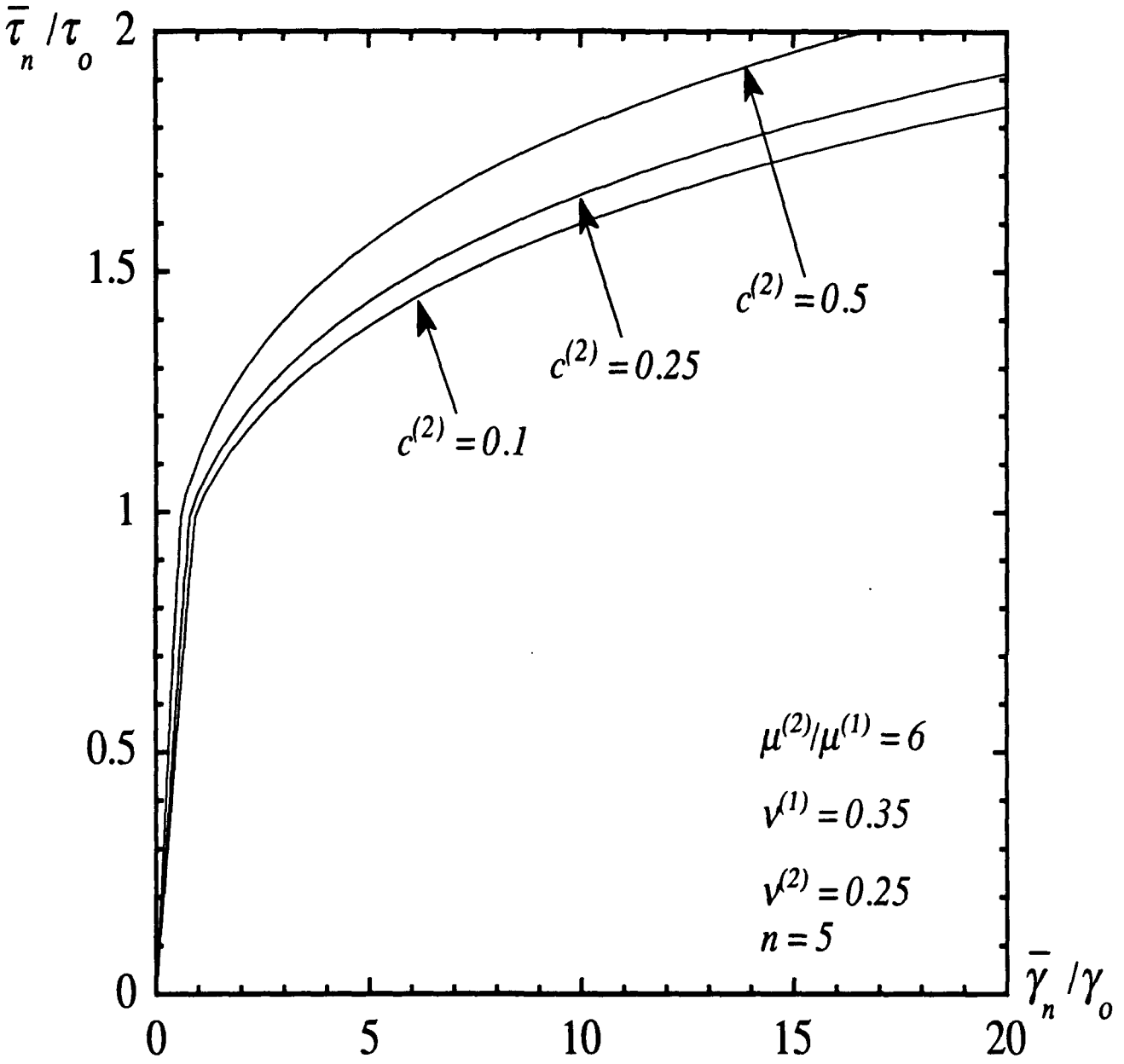
$\bar{\tau}_n/\tau_o$
2
1.5
1
0.5
0
0
5
10
15
20
$\bar{\gamma}_n/\gamma_o$
$c^{(2)}=0.5$
$c^{(2)}=0.25$
$c^{(2)}=0.1$
$\mu^{(2)}/\mu^{(1)}=6$
$\nu^{(1)}=0.35$
$\nu^{(2)}=0.25$
$n=5$

fig 9

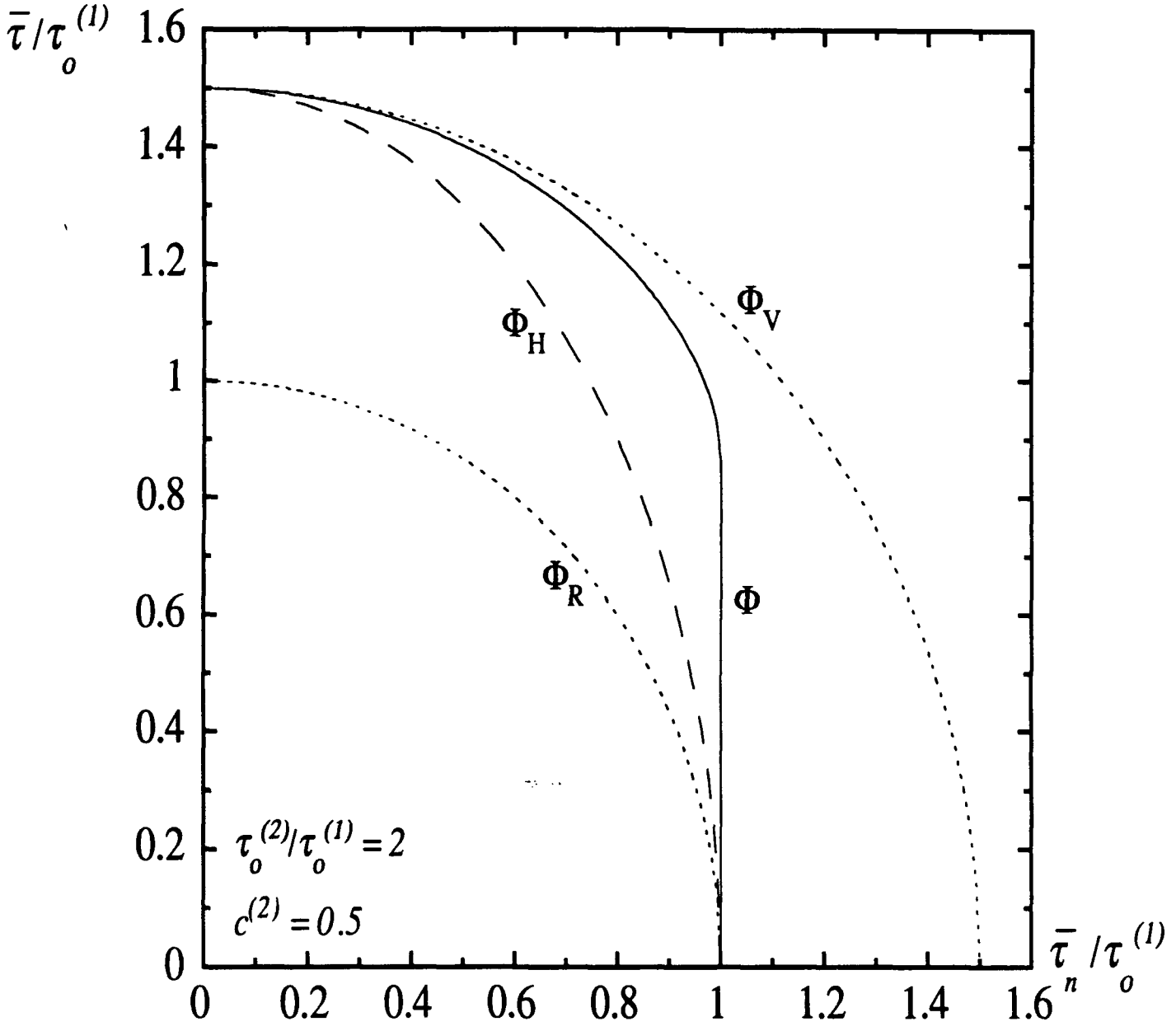
$\bar{\tau}/\tau_o^{(1)}$
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
Φ_V
Φ_H
Φ_R
Φ
$\tau_o^{(2)}/\tau_o^{(1)} = 2$
$c^{(2)} = 0.5$
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
$\bar{\tau}_n/\tau_o^{(1)}$

Fig-10

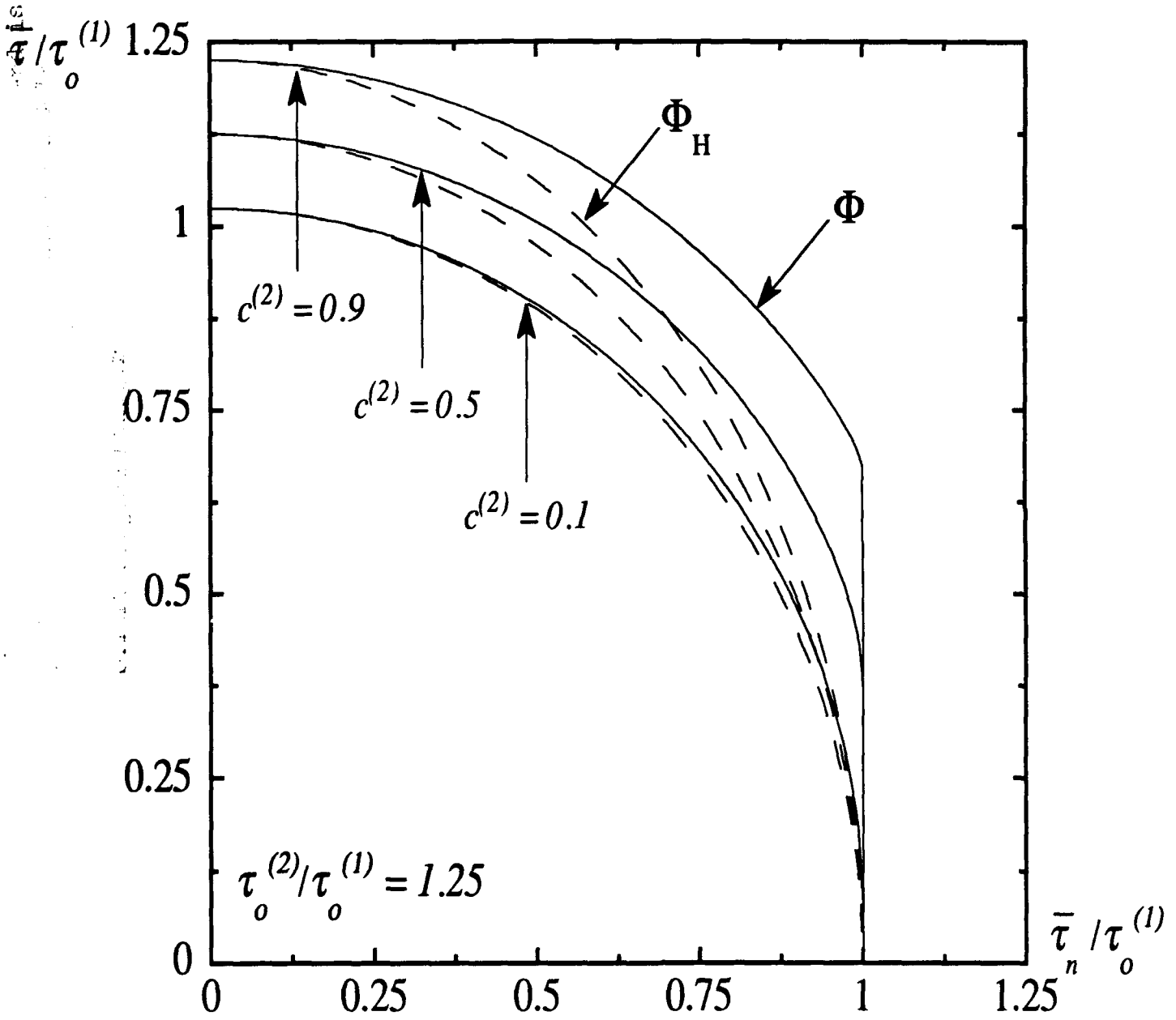
$\bar{\tau}_n / \tau_o^{(1)}$
1.25
1
0.75
0.5
0.25
0
0
0.25
0.5
0.75
1
1.25
$\bar{\tau}_n / \tau_o^{(1)}$
Φ_H
Φ
$c^{(2)} = 0.9$
$c^{(2)} = 0.5$
$c^{(2)} = 0.1$
$\tau_o^{(2)} / \tau_o^{(1)} = 1.25$

Fig-11